Student Number: STUDNUMBER

Statistical Mechanics Subject Exam

December 10th, 2018

Do not write your name on the exam.

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \text{ (for } a > 0)$$

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m} = \frac{1}{\Gamma(m)} \int_0^{\infty} dx \frac{x^{m-1}}{e^x - 1}$$

$$\Gamma(n) = (n-1)! = \int_0^{\infty} dx x^{n-1} e^{-x}$$

$$\ln N! \approx N \ln N - N \text{ (for } N \gg 1)$$

1. Consider a non-interacting, spin zero gas of Bosons with the dispersion relation $\epsilon(p) = pc$, where ϵ is the single-particle energy and c is a constant.

   (a) (10 pts) Find the critical temperature for Bose condensation in three-dimensions in terms of the particle density.

   (b) (5 pts) What is the lower critical dimension for Bose condensation for this gas? Justify your answer.

Extra room for problem 1.

2. For a classical, two-dimensional gas of N distinguishable particles with total energy

$$E = \sum_{i=1}^{N} \left(\frac{\vec{p}_i^2}{2m} + m\frac{\omega^2}{2}\vec{x}_i^2 \right),$$

in the canonical ensemble at a temperature T, calculate the

(a) (10 pts) canonical partition function.

(b) (5 pts) entropy of the gas.

(c) (5 pts) pressure of the gas.

Extra room for problem 2.

3. (a) (10 pts) In the canonical ensemble, show that fluctuations in the energy are given by

$$\langle \Delta E^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 = T^2 C_V. \tag{1}$$

(b) (10 pts) Given a one-dimensional system with an energy density $h(x)$, so that $E = \int dx h(x)$, find the heat capacity of the system at constant volume from the energy density correlation function

$$\langle \delta h(x) \delta h(x') \rangle = A|x - x'| \exp(-|x' - x|/\ell)$$

assuming that the length of the system $L \gg \ell$.

Extra room for problem 3.

4. (5 pts) Prove the relation

$$\left(\frac{\partial S}{\partial \mu}\right)_{T,V} = \left(\frac{\partial N}{\partial T}\right)_{\mu,V}.$$

Extra room for problem 4.

5. (10 pts) Find the critical exponent β for a system with a Landau-Ginzburg free energy near the critical point given by

$$F = \int d^3x \left[\frac{t}{2} m^2 + g m^6 \right]. \tag{2}$$

where m is a scalar field and $t = (T - T_c)/T_c$ where T_c is the critical temperature.

Extra room for problem 5.

6. (10 pts) Consider a system that exhibits a liquid-gas phase transition with a pressure of the form

$$P = \frac{T}{v - v_0} - \frac{a}{v^3}, \tag{3}$$

find the critical temperature T_c, critical specific volume v_c, and critical pressure P_c.

Extra room for problem 6.

Short Answer Section

7. Consider the Ising model and say if each statement is true or false (2 pts each):

   (a) In the exact solution for one-dimension there is no finite temperature phase transition.

   _____

   (b) The critical exponents are the same for the mean field solution and the exact solution for two-dimensions

   _____

   (c) The critical exponents are the same for the mean field solution and the exact solution for four-dimensions

   _____

8. For a D-dimensional low-temperature Fermi gas, indicate the temperature dependence of the leading order contribution to the heat capacity at constant volume. Stated another way, write down α in $C_V \propto T^\alpha$ for (2 pts each)

   (a) $D = 2 \rightarrow \alpha =$ _____

   (b) $D = 3 \rightarrow \alpha =$ _____

9. (2 pts) A gas of two species of particles A and B, which can decay into one another via the reactions $A \leftrightarrow B$, fills a balloon in a room at constant temperature and pressure. If the entropy of the gas can be written as $S(E, N_{tot}, V, X)$ where $N_{tot} = N_A + N_B$ and $X = N_A/N_{tot}$ and the system is allowed to approach chemical equilibrium, which of the following is true?

   (a) The total entropy of the gas is maximized.

   (b) The total Helmholtz free energy of the gas is minimized.

   (c) The total Helmholtz free energy of the gas is maximized.

   (d) The total Gibbs free energy of the gas is minimized.

   (e) The total Gibbs free energy of the gas is maximized.

10. (2 pts) Two systems that belong to the same universality class have the same (circle all that are true)

    - critical exponents

- critical temperature
- interparticle interaction
- order parameter

11. On these P-v axes, plot several isotherms illustrating the qualitative properties of a liquid gas phase transition. Show (1 pt each):

    (a) An isotherm with $T > T_c$.

    (b) An isotherm with $T = T_c$.

    (c) An isotherm with $T < T_c$

    (d) Draw the Maxwell construction for the $T < T_c$ isotherm.

    (e) Label the region of spinodal instability on the $T < T_c$ isotherm.

    (f) Label the critical point.