

Quantum Mechanics
PHY851-PHY852
Subject Exam
Thursday August 31st 2006
Room 1400 BPS ; 06:00 - 09:00 pm

ID number:

- This is a closed book exam and you do not need a calculator
- Report on this page the ID number you have picked up on the signing sheet
- Report also the ID number on each of your answer sheets
- There are five separate problems. Please start each of them on a new page
- Please lay down your answers neatly in such a way that we can easily figure out partial credits

1 Angular momentum algebra

For a particle in a state $|LM\rangle$ with certain values of angular momentum L and its projection M on the quantization axis (z), find the expectation values of the following operators

(a) L_x and L_y . [1 point]

(b) $L_x L_y$; $L_y L_x$ and L_x^2 ; L_y^2 . [1 point]

(c) L_n and the mean square fluctuation

$$\langle(\Delta L_n)^2\rangle = \langle L_n^2\rangle - \langle L_n\rangle^2. \quad (1)$$

where $L_n = \vec{L} \cdot \vec{n}$ is the projection of the angular momentum operator \vec{L} onto the axis whose unit vector \vec{n} is defined by the polar angle θ and the azimuthal angle φ . [1.5 points]

2 Tensor operator

Consider two spin-1/2 particles, whose spins are described by the Pauli operators $\vec{\sigma}_1$ and $\vec{\sigma}_2$. The axis of spin quantization is, as usual, taken to be the z -axis. Let \vec{e}_r be the unit vector connecting the two particles along the direction defined by $\vec{r} = \vec{r}_1 - \vec{r}_2$. We introduce the tensor operator:

$$S_{12} = 3(\vec{\sigma}_1 \cdot \vec{e}_r)(\vec{\sigma}_2 \cdot \vec{e}_r) - \vec{\sigma}_1 \cdot \vec{\sigma}_2. \quad (2)$$

- (a) Let us consider that the unit vector \vec{e}_r is aligned with the z axis. Show that, if the two particles occupy a spin-singlet state $|X_{singlet}\rangle$, one has $S_{12}|X_{singlet}\rangle = 0$. [1 point]
- (b) In the same situation, show that, if the two particles occupy a spin-triplet state $|X_{triplet}\rangle$, one has $(S_{12} - 2)(S_{12} + 4)|X_{triplet}\rangle = 0$. [1.5 points]
- (c) Consider now that the unit vector \vec{e}_r is not aligned with the z axis and that its orientation is defined with respect to the z axis through the polar and azimuthal angles (θ, φ) . Write down S_{12} in terms of $(\sigma_1)_{x,y,z}$, $(\sigma_2)_{x,y,z}$ and (θ, φ) . Sketch the steps of the calculations you would take to calculate $S_{12}|X_{singlet}\rangle$ and $S_{12}|X_{triplet}\rangle$ in this case. [1 point]

3 1D attractive potential and its bound state(s)

Consider a 1D potential $V(x)$ which is attractive in a portion of the real axis and never positive. Use the variational method to prove that such a potential always has, at least, one bound state. You can use a convenient trial wave function such as $\psi(x) = Ne^{-\beta^2 x^2/2}$. [3 points]

Note:

$$\int_{-\infty}^{+\infty} dx e^{-x^2} = \sqrt{\pi} ; \int_{-\infty}^{+\infty} dx x^2 e^{-x^2} = \frac{\sqrt{\pi}}{2} . \quad (3)$$

4 Multi-center scattering

A particle of mass m and incoming momentum $\vec{p} = \hbar\vec{k}$ is elastically scattered off a planar molecule made of four identical heavy atoms located at the corners of a square with a side a . The square is located in the (x, y) plane and is centered at the origin. The particle is scattered into a state of momentum $\vec{p}' = \hbar\vec{k}'$. The interaction potential between the particle and one of the atom is given by $U(\vec{r} - \vec{b})$, where \vec{b} is the position vector of the atom. In the Born approximation, one can write the scattering amplitude from a single atom located at the origin and interacting with the incoming particle of mass m via the potential $U(\vec{r})$ under the form

$$f_0(\vec{q}) = -\frac{m}{2\pi\hbar^2} \int d\vec{r} e^{i\vec{q}\cdot\vec{r}} U(\vec{r}). \quad (4)$$

where $\vec{q} = \vec{k}' - \vec{k}$ is the so-called *momentum transfer*¹ vector.

- Find, in the Born approximation, the relation between the scattering amplitudes, differential and total cross sections for the scattering off the molecule and the scattering off an individual atom located at the origin. [2 points]
- Find the coordinates of the momentum transfer (q_x, q_y) corresponding to the maxima of scattering. [2 point]
- Relate q to k and the scattering angle θ . [1 point]
- Discuss the low and high energy cases. [2 points]

¹This is an abuse of language since, \vec{q} , \vec{k} and \vec{k}' are wave vectors and are related to the corresponding momenta through a factor \hbar .

5 Free fermions in a box

We consider N free fermions in a box of volume V and do not worry about their spin in the present problem. We choose to work with the plane-wave single-particle basis:

$$\varphi_{\vec{k}}(\vec{r}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}}, \quad (5)$$

where the wave vector \vec{k} is quantized due to the use of periodic boundary conditions in the box. Write down the second quantized form of the one-body density operator $\hat{\rho}(\vec{r}) = \sum_{i=1}^N \delta(\vec{r} - \hat{r}_i)$. [3 points]