Quantum Mechanics
PHY851-PHY852
Subject Exam
Wednesday May 3^{rd} 2006
Room 1400 BPS ; 7:45 - 10:45 am

Student number:

- This is a closed book exam and you do not need a calculator
- Report the student number you have picked up on this page
- Report also the student number on each of you answer sheets
- There are five separate problems. Please start each of them on a new page
- Please lay down your answers neatly in such a way that we can easily figure out partial credits

1 Particles with spin in a magnetic field

The Hamiltonian of a charged particle (charge q) with spin $1/2$ interacting with the electromagnetic field $(\vec{E}, \vec{B})$ is

$$\hat{H} = \frac{\left(\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\right)^2}{2m} + q\phi(\hat{\vec{r}}, t) - \hat{\vec{\mu}} \cdot \vec{B}(\hat{\vec{r}}, t) \tag{1}$$

where $\hat{\vec{\mu}} = \mu_0 \hat{\vec{\sigma}}$ and μ_0 are the spin magnetic moment operator and the spin magnetic moment of the particle, respectively. $\phi(\hat{\vec{r}}, t)$ and $\vec{A}(\hat{\vec{r}}, t)$ denote the scalar and vector potentials, respectively.
(a) Find the stationary states and energy levels of a free neutral spin-1/2 particle in a constant and homogeneous magnetic field oriented along the z-axis $\vec{B} = B\,\vec{e}_z$. $[2\ points]$
(b) Show that the Hamiltonian of an electron in the electromagnetic field $(\vec{E}, \vec{B})$ can be written as

$$\hat{H} = \frac{\left\{\hat{\vec{\sigma}} \cdot \left(\hat{\vec{p}} + e\vec{A}(\hat{\vec{r}}, t)\right)\right\}^2}{2m} - e\,\phi(\hat{\vec{r}}, t) \tag{2}$$

$[1\ point]$
(c) Consider an electron in a constant magnetic field. Show that the projection of the spin of the electron along the direction of its velocity is a constant of motion. $[1\ point]$

2 Zero-range interaction in 1D

Two particles with the same mass m are confined to one dimension (1D) and interact via the potential $U(x_1 - x_2)$.
(a) Separate out the center of mass motion from the relative motion in the Hamiltonian. Write the Schrödinger equation for the relative motion in terms of the reduced mass μ. $[1\ point]$
(b) We now consider instead of $U(x_1 - x_2)$, the potential $V(x) = -v_0\,\delta(x)$ with $v_0 > 0$, where x denotes the relative position vector. Determine the bound-state normalized wave function for the relative motion and the corresponding binding energy. $[1\ point]$
(c) Consider that the original realistic interaction $U(x)$ sustains only one bound state and that the energy of the bound state (E) is much smaller than the strength of the interaction, i.e. $|E| \ll \max|U(x)|$. Show that the $U(x)$ can be replaced by $V(x) = -v_0\,\delta(x)$, where $v_0 = -\int_{-\infty}^{+\infty} U(x)dx$, if the range of $U(x)$ is much smaller than the region in which the bound-state wave function changes significantly. $[2\ points]$
(d) Using (b) and (c), derive an expression for the energy of the bound system as a function of $U(x)$ in the zero-range approximation. $[1\ point]$

3 Spin-dependent scattering process

The scattering amplitude for neutron-proton scattering takes the form

$$\hat{f}_E(\theta)\,|P:\sigma;N:\sigma'\rangle = \left(A(E,\theta)+B(E,\theta)\,\vec{\sigma}_P\cdot\vec{\sigma}_N\right)\,|P:\sigma;N:\sigma'\rangle \tag{3}$$

where $\hat{f}_E(\theta)$ is an operator in spin space, $\hat{f}_E(\theta)\,|P:\sigma;N:\sigma'\rangle$ is the spin part of the neutron-proton scattering state. The basis used for the two-body spin state is constituted by the four states obtained by taking the product of the neutron and proton spinors

$$|P:\sigma;N:\sigma'\rangle \;\;=\;\; \chi^P_\sigma\,\chi^N_{\sigma'} \tag{4}$$

where $\chi^q_{\uparrow/\downarrow}$ is the spin-up/spin-down state of the particle q.
(a) Calculate the four scattering amplitudes in the basis defined by Eq. (4). $[1.5\;points]$
(b) The differential cross section is then defined for a each spin channel. Those channels correspond to the 16 possible combinations of initial and final spin states in the scattering process. In the basis given by Eq. (4), the differential cross section for one channel is defined as

$$\frac{d\sigma}{d\Omega}\,(\sigma\,\sigma'\to\sigma''\,\sigma''') = \left|f_E(\theta;\sigma\,\sigma'\to\sigma''\,\sigma''')\right|^2 = \left|\langle P:\sigma'';N:\sigma'''|\,\hat{f}_E(\theta)\,|P:\sigma;N:\sigma'\rangle\right|^2 \tag{5}$$

Tabulate the 16 possible differential cross sections induced by $\hat{f}_E(\theta)$, in Eq. (3), under a matrix form. $[1.5\;points]$
(c) Using the previous results, calculate the cross sections for *triplet* $\rightarrow$ *triplet* and *single* $\rightarrow$ *singlet* scattering, respectively. Show that the *triplet* $\rightarrow$ *singlet* cross section vanishes. $[1\;point]$

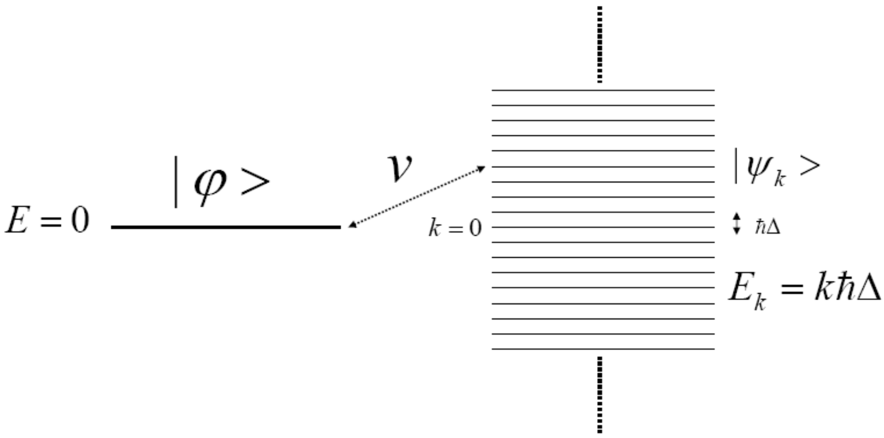

Figure 1: Spectrum of H_0. The states $|\varphi\rangle$ represents a quasi bound state with energy $E = 0$, whereas $|\psi_k\rangle$ denotes an infinite set of equally spaced states with energy $E_k = k\,\hbar\Delta$, where k is a positive or a negative integer.

4 Coupling to a "structured" continuum of states

First, we consider the unperturbed Hamiltonian H_0 defined by:

$$\begin{aligned} H_0\,|\varphi\rangle &= E\,|\varphi\rangle = 0 \\ H_0\,|\psi_k\rangle &= E_k\,|\psi_k\rangle = k\,\hbar\,\Delta\,|\psi_k\rangle \end{aligned}$$

where k is a positive or a negative integer. The small parameter Δ controls the energy spacing between the states $\{|\psi_k\rangle\}$. The situation is graphically depicted in Fig. 1.
Second, we consider the time-independent coupling V between the eigenstates of H_0 defined by:

$$\begin{aligned} \langle\varphi|\,V\,|\varphi\rangle &= \langle\psi_k|\,V\,|\psi_{k'}\rangle = 0 \qquad \text{for all } (k,k') &(6)\\ \langle\psi_k|\,V\,|\varphi\rangle &= \langle\varphi|\,V\,|\psi_k\rangle = v \qquad \text{for all } k\ , &(7) \end{aligned}$$

and study the decay of the initial state $|\varphi\rangle$ to the continuum of final states $\{|\psi_k\rangle\}$ due to that coupling V.
(a) The transition probability per unit time from $|\varphi\rangle$ to $|\psi_k\rangle$ due to a harmonic coupling $W(t) = W\,e^{\mp i\omega t}$ is given, from first order time-dependent perturbation theory, as:

$$\Gamma_k = \frac{2\pi}{\hbar}\,|\langle\psi_k|\,W\,|\varphi\rangle|^2\,\delta(E_k - E \mp \hbar\omega)\ . \tag{8}$$

Calculate Γ_k in the present case in terms of v and Δ. [2 *points*]
(b) Calculate the total decay probability per unit time Γ by summing Γ_k over all final states[1]. Give the expression of Γ as a function of v and Δ. [1 *point*]
(c) Identify in the previous result the density of final states which usually enters the Fermi Golden rule. Explain its value from the properties of the energy spectrum. [1 *point*]

[1]Since the system decays to a dense continuum, the sum over final states becomes an integral. To do that properly, one must perform the replacement $\sum_k \longrightarrow \int_{-\infty}^{+\infty} dk/N(k)$, where $N(k)$ is the number of values of k per unit interval in the original sum. This number can evolve with k apriori. Here, $N(k) = 1$ since k takes integer values.

5 Free fermions in a box

We consider N free fermions in a box of volume V and do not worry about their spin in the present problem. We choose to work with the plane-wave single-particle basis:

$$\varphi_{\vec{k}}(\vec{r}) = \frac{1}{\sqrt{V}}\, e^{i\vec{k}\cdot\vec{r}} \quad , \tag{9}$$

where the wave vector k is quantized due to the use of periodic boundary conditions in the box. Write down the second quantized form of the one-body density operator $\hat{\rho}(\vec{r}) = \sum_{i=1}^{N} \delta(\vec{r} - \hat{\vec{r}}_i)$. $[3\ points]$