PHY-841: Classical Electrodynamics / Subject Exam / Apr 29, 2021

Please read all of the following before starting the exam:

- Please access the exam problems on Gradescope only after we agree to start *in the Zoom meeting*
- Please stay connected to Zoom during the entire duration of your exam and switch on your camera.
- You have 3 hours to solve the problems.
- Please do *not* write your name on your solution sheets.
- Please upload your solutions to Gradescope and indicate for each problem where the relevant pages are.
- To solve the problems, you may use a simple calculator, but no computer algebra systems, external notes, books, web searches etc.
- All problems are in S.I. units unless stated otherwise. Please give your answers in terms of the given variables and units.
- A complete answer usually includes a derivation of the result (unless stated otherwise). Show all work as neatly and logically as possible to maximize your credit. State clearly which equations were used. Circle or otherwise indicate your final answers.
- Please ask if the problem description is unclear.
- *Good luck !*

$$
\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b}),\\
\vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b}),\\
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}),\\
\vec{\nabla}\times(\vec{\nabla}\psi) &= 0,\\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) &= 0,\\
\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{a})-\nabla^2\vec{a},\\
\vec{\nabla}\cdot(\psi\vec{a}) &= \vec{a}\cdot\vec{\nabla}\psi+\psi\vec{\nabla}\cdot\vec{a},\\
\vec{\nabla}\times(\psi\vec{a}) &= \vec{\nabla}\psi\times\vec{a}+\psi\vec{\nabla}\times\vec{a},\\
\vec{\nabla}(\vec{a}\cdot\vec{b}) &= (\vec{a}\cdot\vec{\nabla})\vec{b}+(\vec{b}\cdot\vec{\nabla})\vec{a}+\vec{a}\times(\vec{\nabla}\times\vec{b})+\vec{b}\times(\vec{\nabla}\times\vec{a}),\\
\vec{\nabla}\cdot(\vec{a}\times\vec{b}) &= \vec{b}\cdot(\vec{\nabla}\times\vec{a})-\vec{a}\cdot(\vec{\nabla}\times\vec{b}),\\
\vec{\nabla}\times(\vec{a}\times\vec{b}) &= \vec{a}(\vec{\nabla}\times\vec{b})-\vec{b}(\vec{\nabla}\times\vec{a})+(\vec{b}\cdot\vec{\nabla})\vec{a}-(\vec{a}\cdot\vec{\nabla})\vec{b},\\
\vec{\nabla}\cdot\vec{r} &= 3,\\
\vec{\nabla}\times\vec{r} &= 0,\\
\vec{\nabla}\cdot\hat{r} &= 2/r,\\
\vec{\nabla}\times\hat{r} &= 0,\\
\vec{\nabla}r &= \hat{r},\\
\vec{\nabla}\frac{1}{r} &= -\frac{\hat{r}}{r^2},\\
\vec{\nabla}\cdot(\hat{r}f(r)) &= \frac{2}{r}f+\frac{df}{dr},\\
(\vec{a}\cdot\vec{\nabla})\hat{r} &= \frac{1}{r}[\vec{a}-\hat{r}(\vec{a}\cdot\hat{r})]=\frac{\vec{a}_\perp}{r},\\
\vec{\nabla}^2\left(\frac{1}{r}\right) &= -4\pi\delta(\vec{r}),\\
\int_V d^3r\ \vec{\nabla}\cdot\vec{A} &= \int_S d\vec{S}\cdot\vec{A},\\
\int_V d^3r\ \vec{\nabla}\psi &= \int_S \psi\ d\vec{S},\\
\int_V d^3r\ \vec{\nabla}\times\vec{A} &= \int_S d\vec{S}\times\vec{A},\\
\int_V d^3r\ (\phi\nabla^2\psi+\vec{\nabla}\phi\cdot\vec{\nabla}\psi) &= \int_S \phi\ d\vec{S}\cdot\vec{\nabla}\psi,\\
\int_V d^3r\ (\phi\nabla^2\psi-\psi\nabla^2\phi) &= \int_S (\phi\vec{\nabla}\psi-\psi\vec{\nabla}\phi)\cdot d\vec{S},\\
\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} &= \oint d\vec{\ell}\cdot\vec{A},\\
\int_S d\vec{S}\times\vec{\nabla}\psi &= \oint_C d\vec{\ell}\ \psi.
\end{aligned}
$$

$$\mathscr{L} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu$$
$$L = \frac{1}{\gamma}(-mc^2 - qA_\mu u^\mu)$$
$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu$$
$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$
$$\partial_\mu J^\mu = 0$$
$$(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1),$$
$$\vec{\beta} = \vec{v}/c,$$
$$\gamma = 1/\sqrt{1-\beta^2},$$
$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$
$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x^\mu = (ct, x, y, z),$$
$$\partial^\mu = ((1/c)\partial/\partial t, -\vec{\nabla})$$
$$k^\mu = (\omega/c, \vec{k}),$$
$$u^\mu = (\gamma c, \gamma\vec{v}),$$
$$p^\mu = (E/c, \vec{p}),$$
$$A^\mu = (\phi/c, \vec{A}),$$
$$J^\mu = (c\rho, \vec{j}),$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$
$$\vec{E}'_\parallel = \vec{E}_\parallel$$
$$\vec{B}'_\parallel = \vec{B}_\parallel$$
$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + c\vec{\beta}\times\vec{B})$$
$$\vec{B}'_\perp = \gamma(\vec{B}_\perp - \frac{1}{c}\vec{\beta}\times\vec{E})$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho, \qquad \vec{\nabla}\cdot\vec{B} = 0,$$
$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{j}, \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0, \qquad \epsilon_0\mu_0 = \frac{1}{c^2}$$

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}), \qquad 0 = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}, \qquad \vec{j}(\vec{r},t) = q\vec{v}(t)\,\delta(\vec{r}-\vec{r}_0(t)), \qquad I = \frac{dq}{dt}$$
$$\vec{B} = \vec{\nabla}\times\vec{A}, \qquad I = \int_S \vec{j}\cdot d\vec{A}, \qquad I d\vec{l} = dq\,\vec{v}$$

$$(\vec{E}_{\mathrm{out}} - \vec{E}_{\mathrm{in}})\cdot\hat{n} = \sigma/\epsilon_0$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\rho(\vec{r}')G(\vec{r},\vec{r}') - \frac{1}{4\pi}\int_S dA'\phi(\vec{r}')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}$$
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\rho(\vec{r}')G(\vec{r},\vec{r}') + \frac{1}{4\pi}\int_S dA'\frac{\partial\phi(\vec{r}')}{\partial n'}G(\vec{r},\vec{r}') + \langle\phi\rangle_S$$
$$\phi = \sum_{l=0}^{\infty}\left(a_l r^l + b_l/r^{l+1}\right)P_l(\cos\theta)$$
$$\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}/r^{l+1}\right]Y_{lm}(\theta,\varphi)$$
$$\phi = a_0 + b_0\ln s + \sum_{n=1}^{\infty}\left[s^n(a_n\cos(n\varphi) + b_n\sin(n\varphi)) + s^{-n}(c_n\cos(n\varphi) + d_n\sin(n\varphi))\right]$$

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$
$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$
$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$
$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right), \quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}, \quad Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$
$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}e^{im\varphi}P_l^m(\cos\theta)$$
$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$$
$$\phi = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{q_{lm}}{r^{l+1}}Y_{lm}(\theta,\varphi)$$
$$q_{lm} = \int d^3r'\,\rho(\vec{r}')r'^l Y_{lm}^*(\theta,\varphi)$$
$$\phi = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\text{tot}}}{r} + \frac{p_i\hat{r}_i}{r^2} + \frac{1}{2!}Q_{ij}\frac{\hat{r}_i\hat{r}_j}{r^3} + \dots\right)$$
$$p_i = \int d^3\,r'\rho(\vec{r}')\,r_i'$$
$$Q_{ij} = \int d^3r'\,\rho(\vec{r}')\,(3r_i'r_j' - \delta_{ij}r'^2)$$
$$U = Q_{\text{tot}}\phi(\vec{r}) - p_iE_i(\vec{r}) - \frac{1}{6}Q_{ij}\partial_iE_j + \dots$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{\vec{m}\times\vec{r}}{r^3} + \dots\right)$$
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} + \dots\right)$$
$$\vec{m} = \frac{1}{2}\int d^3r'\,\vec{r}'\times\vec{j}(\vec{r}')$$
$$U = -m_iB_i(\vec{r}) + \dots$$
$$\vec{\tau} = \vec{m}\times\vec{B} + \vec{r}\times\vec{F} + \dots$$

$$\vec{A} = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\left(\dot{\vec{p}}\right) + \frac{1}{rc}\left(\dot{\vec{m}} \times \hat{r} + \frac{1}{6}\ddot{\vec{Q}}\right) + \ldots\right]$$

$$\vec{B} = \frac{\mu_0}{4\pi c}\left[\frac{1}{r}\left(\ddot{\vec{p}} \times \hat{r}\right) + \frac{1}{rc}\left((\ddot{\vec{m}} \times \hat{r}) \times \hat{r} + \frac{1}{6}\dddot{\vec{Q}} \times \hat{r}\right) + \ldots\right]$$

$$\vec{E} = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\left((\ddot{\vec{p}} \times \hat{r}) \times \hat{r}\right) + \frac{1}{rc}\left(\ddot{\vec{m}} \times \hat{r} + \frac{1}{6}(\dddot{\vec{Q}} \times \hat{r}) \times \hat{r}\right) + \ldots\right]$$

$$\frac{dP}{d\Omega} = |\vec{S}|r^2 = \frac{c}{\mu_0}|\vec{B}|^2 r^2$$

$$P(t) = \frac{\mu_0}{4\pi c}\left(\frac{2}{3}|\ddot{\vec{p}}|^2 + \frac{2}{3c^2}|\ddot{\vec{m}}|^2 + \frac{1}{180c^2}\dddot{Q}_{ij}\dddot{Q}_{ji}\right)$$

$$\vec{B} = i\vec{k} \times \vec{A}$$

$$\vec{E} = c\vec{B} \times \hat{r}$$

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \qquad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + c\vec{\beta} \times \vec{B})$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}, \qquad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \frac{1}{c}\vec{\beta} \times \vec{E})$$

$$u = \frac{1}{2}\left(\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right)$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

$$0 = \vec{E} \cdot \vec{j} + \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S}$$

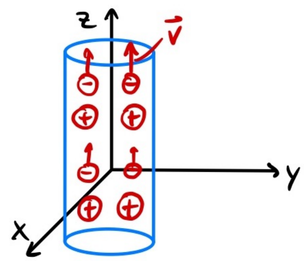
$$\vec{g} = \epsilon_0 \vec{E} \times \vec{B}$$

$$\sigma_{ij} = \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}|\vec{E}|^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}|\vec{B}|^2\right)$$

$$0 = \left(\rho\vec{E} + \vec{j} \times \vec{B} + \frac{\partial}{\partial t}\vec{g}\right)_j - \frac{\partial}{\partial x_i}\sigma_{ij}$$

1 Charges and currents

Consider a current-carrying long cylinder of radius R oriented along the z axis. The cylinder contains negative charges, which move with an average velocity $\vec{v} = v\hat{z}$ inside the cylinder and generate a uniform charge density ρ_- in the lab frame. In addition, there are positive charges with uniform charge density ρ_+, these are at rest with respect to the lab frame.



1.a [20 pts] Derive the magnetic field and the *radial component* of the electric field at a distance s $(0 < s < R)$ from the z axis (center of the cylinder).

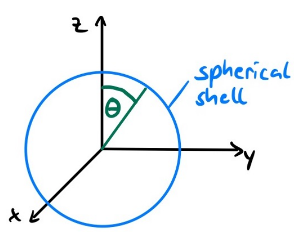
1.b [5 pts] Consider a charge q moving with $\vec{v} = v\hat{z}$ (same v as for negative charges generating ρ_-) at a distance r $(0 < r < R)$ from the center of the cylinder. Assume the net radial force on the charge q to be zero, allowing for a uniform steady current in the cylinder. Derive the relation betwen ρ_+ and ρ_- required to achieve this.

2 Charged hollow sphere

[25 pts] A hollow sphere of radius R is centered at the origin. The (non-conducting) surface of the sphere is charged with density

$$\sigma = \sigma_0 \left(1 - \frac{3}{2}\sin^2\theta\right) \tag{1}$$

where σ_0 is constant and θ is the polar angle.



Determine the electrostatic potential *inside* and *outside* of the sphere.

3 Time ordering

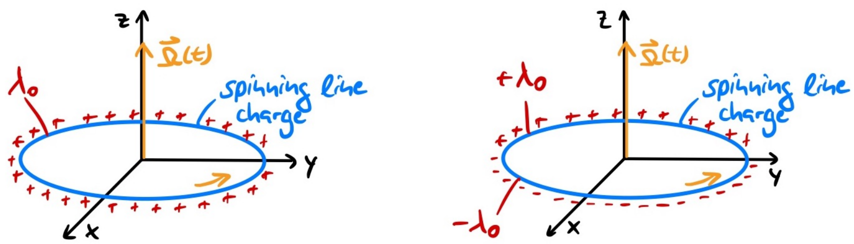
[5 pts] Two alarm clocks are separated by a distance of 10^6 m. If one goes off Δt after the other in their rest frame, will that time ordering always be the same if observed from a frame moving with some constant velocity ? If not, give a specific condition on Δt for which the time ordering is unambiguous in *any* inertial frame.

4 Oscillating line charge

A charged ring with radius R is mounted to an insulating disc. The disc lies in the $x-y$ plane, is centered at the origin, and rotates around the z axis. The angular velocity of the disc $\vec{\Omega}$ oscillates with time,

$$\vec{\Omega} = \Omega_0 \hat{z} \cos(\omega t), \tag{2}$$

where $\omega \ll \Omega \ll c/R$.



4.a [15 pts] Consider a constant line charge $\lambda = \lambda_0$ for the spinning ring (left figure). Derive the average energy per time the system will lose due radiation (keep only the leading, non-zero contribution in the multipole approximation).

4.b [5 pts] Derive the angular power spectrum of the radiation for an observer far away from the charges. Let $\vec{r} = r(\sin\theta, 0, \cos\theta)$ be the observer position.

4.c [10 pts] Compare the situation to a case, where half of the ring is charged with λ_0 and the other half with $-\lambda_0$ (right figure). Discuss *qualitatively* possible changes in the angular dependence of the radiation. Comment in particular on radiation in the $\hat{x}$ and $\hat{z}$ direction.

5 Moving light source

A light source moves away from an observer at a speed of $0.8c$. The source emits a light pulse, which propagates through space and is detected by the observer with a wave length of 666 nm. In the observer frame, the light source is at a distance of 10^{15} m when it emits the pulse.

5.a [5 pts] In the rest frame of the light source, how long does it take the pulse to reach the observer ?

5.b [5 pts] In the rest frame of the light source, what is the wave length of the emitted light ?

6 Acceleration of a space ship

[5 pts] Is it possible, at least in principle, to construct a device, which accelerates a space ship in free space using only electromagnetic fields ? Discuss conditions on the electric and magnetic fields for the area A covering the region behind the space-ship where the engine is located (see figure), which need to be fulfilled for this kind of engine to work.