Student Number:______________________________

Electricity & Magnetism **Subject Exam** **May 1, 2014**

Please read all of the following before starting the exam:

- Before starting the exam, write your student number on each page of the exam. If you require extra paper, write your student number and the relevant problem number on the extra page(s).
- All problems are assumed to be in Gaussian units. If you choose to convert to SI units, please state so. You will be responsible for the correct conversion factors.
- You may use a simple calculator, but no external notes, books, etc.
- Show all work as neatly and logically as possible to maximize your credit. Circle or otherwise indicate your final answers.
- This test has 5 problems for a total of 100 points. Please make sure that you have all of the pages.
- Good luck!

VECTOR CALCULUS

Gradient vector $= \vec{\nabla} = (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$
$\mathrm{div}\,\mathbf{v} \equiv \vec{\nabla}\cdot\mathbf{v}$
$\mathrm{curl}\,\mathbf{v} \equiv \vec{\nabla}\times\mathbf{v}$

Cylindrical coordinates (ρ, ϕ, z):

$$\nabla S = \frac{\partial S}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial S}{\partial \phi}\mathbf{e}_\phi + \frac{\partial S}{\partial z}\mathbf{e}_z$$

$$\vec{\nabla}\cdot\mathbf{v} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\begin{aligned}\vec{\nabla}\times\mathbf{v} &= \left[\frac{1}{\rho}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\mathbf{e}_\rho \\ &\quad + \left[\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho}\right]\mathbf{e}_\phi \\ &\quad + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial \phi}\right]\mathbf{e}_z\end{aligned}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical polar coordinates (r, θ, ϕ):

$$\nabla S = \frac{\partial S}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\mathbf{e}_\phi$$

$$\vec{\nabla}\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned}\vec{\nabla}\times\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\mathbf{e}_r \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\mathbf{e}_\theta \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\mathbf{e}_\phi\end{aligned}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Spherical Harmonics:

$$l = 0: \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1: \qquad Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$l = 2: \qquad Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$$Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

Legendre Polynomials:

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\; Y_{l,0}(\theta,\phi)\ .$$

SELECTED NUMERICAL DATA

Speed of light $c = 3 \times 10^{10}\,\mathrm{cm/s}$,
Elementary charge $e = 4.8 \times 10^{-10}$ statC,
Planck constant $\hbar = h/2\pi = 1.055 \times 10^{-27}\,\mathrm{erg \cdot s} = 6.582 \times 10^{-22}\,\mathrm{MeV \cdot s}$.
Do not use these numbers directly! Instead combine your expressions in the standard combinations:
Fine structure constant (dimensionless) $\alpha = e^2/\hbar c$, $1/\alpha = 137.036$;
$\hbar c = 197.3\,\mathrm{MeV \cdot fm} \approx 2 \times 10^{-5}\,\mathrm{eV \cdot cm}$ ($1\,\mathrm{fm} = 10^{-13}\,\mathrm{cm}$).

Electron mass $m = 0.911 \times 10^{-27}\,\mathrm{g} = 0.511\,\mathrm{MeV}/c^2$,
Proton mass $m_p = 1.673 \times 10^{-24}\,\mathrm{g} = 938.3\,\mathrm{MeV}/c^2 = 1836.2\,m$,

Compton wave length of the electron $\lambda_e = \hbar/mc = 3.862 \times 10^{-11}\,\mathrm{cm}$,
Classical electron radius $r_e = e^2/mc^2 = 2.818 \times 10^{-13}\,\mathrm{cm}$.

Bohr magneton $\mu_B = e\hbar/2mc = 9.274 \times 10^{-21}\,\mathrm{erg/Gs}$,
Nuclear magneton (n.m.) $\mu_N = e\hbar/2m_p c = \mu_B(m/m_p) = 5.051 \times 10^{-24}\,\mathrm{erg/Gs}$,
Proton magnetic moment $\mu_p = 2.793$ n.m.,
Neutron magnetic moment $\mu_n = -1.913$ n.m.

Gravitational constant $G = 6.67 \times 10^{-8}\,\mathrm{cm^3 g^{-1} s^{-2}}$.

Some conversions between SI and Gaussian units:

$$1\,\mathrm{J} = 10^7\,\mathrm{erg}$$

$$1\,\mathrm{Watt} = 10^7\,\mathrm{erg/s}$$

$$1\,\mathrm{Coulomb} = 3 \times 10^9\,\mathrm{statC}$$

$$1\,\mathrm{Ampere} = 3 \times 10^9\,\mathrm{statA}$$

$$1\,\mathrm{Volt} = \frac{1}{300}\,\mathrm{statV}$$

$$1\,\mathrm{Tesla} = 10^4\,\mathrm{Gs}$$

$$1\,\mathrm{eV}/c^2 = 1.783 \times 10^{-36}\,\mathrm{kg} = 1.783 \times 10^{-33}\,\mathrm{g}$$

Conversion of Maxwell Equations from Gaussian to SI units:

$$(\rho, \mathbf{j}, q) \Rightarrow \frac{(\rho, \mathbf{j}, q)}{\sqrt{4\pi\epsilon_0}},$$

$$(\phi, \mathbf{E}) \Rightarrow \sqrt{4\pi\epsilon_0}\,(\phi, \mathbf{E}),$$

$$(\mathbf{A}, \mathbf{B}) \Rightarrow \sqrt{\frac{4\pi}{\mu_0}}\,(\mathbf{A}, \mathbf{B}),$$

$$c = \sqrt{\frac{1}{\epsilon_0 \mu_0}}.$$

Student Number:________________________________

1. A uniformly charged cylinder of radius R, height h, and total charge Q is centered at the origin, with its symmetry axis along the $\hat{z}$ axis and with $-h/2 \leq z \leq h/2$.

   (a) [**12 pts**] Obtain the first two non-zero terms in the multi-pole expansion for the electrostatic potential, $\Phi(r, \theta, \phi)$.

   (b) [**8 pts**] Obtain the first two non-zero terms in the multi-pole expansion for the electric field, $\mathbf{E}(r, \theta, \phi)$.

Student Number:__________________________

2. Two point charges Q_1 and Q_2 move at constant velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$ and are separated by a distance R in the y-direction. Their trajectories can be written $\mathbf{r}_1(t) = (vt, 0, 0)$ and $\mathbf{r}_2(t) = (vt, R, 0)$ in the laboratory frame K.

   (a) **[5 pts]** Write down the electric and magnetic fields $\mathbf{E}_1'(\mathbf{r}_2')$ and $\mathbf{B}_1'(\mathbf{r}_2')$ due to charge Q_1 (and at the position of charge Q_2) in the frame K' in which both Q_1 and Q_2 are at rest.

   (b) **[5 pts]** Perform the Lorentz transformations on the fields to obtain $\mathbf{E}_1(\mathbf{r}_2)$ and $\mathbf{B}_1(\mathbf{r}_2)$ in the laboratory frame K.

   (c) **[5 pts]** Obtain the force on charge Q_2 due to charge Q_1 in the laboratory frame.

   (d) **[5 pts]** Now suppose that the charges are separated along the $\hat{\mathbf{x}}$ direction, so that $\mathbf{r}_1(t) = (vt, 0, 0)$ and $\mathbf{r}_2(t) = (vt + R, 0, 0)$ in the laboratory frame K. What is the force on charge Q_2 due to charge Q_1 in the laboratory frame for this configuration?

Student Number:________________________

3. A spherical shell of radius R is uniformly charged over its surface with a total charge $Q < 0$. Its total mass is M, which is also uniformly spread over its surface. The shell is rotating around an axis through its center with angular velocity $\boldsymbol{\omega}$.

   (a) [**10 pts**] Calculate the magnetic moment of the shell around its rotation axis.

   (b) [**5 pts**] The shell is placed in a constant uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$. At time $t = 0$, the rotation axis is along the $\hat{\mathbf{x}}$ direction. Describe the subsequent motion of the shell. At what time does the shell return to its initial condition?

   (c) [**4 pts**] Now suppose that the shell also has an initial center-of-mass velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$ (non-relativistic) at $t = 0$. Describe the orbital motion of the shell. What is its radius and period?

   (d) [**1 pt**] At what times does the shell return to its exact initial conditions (both magnetic moment and orbital)?

   (e) [**Extra Credit: 1 pt**] How would your answer to part (d) be different if the spherical shell were replaced by an electron?

4. The Large Hadron Collider is a proton-proton collider. However, the high-energy processes can be effectively described as the scattering of constituents of the proton (quarks or gluons), which carry a fraction of the proton energy. Thus, the Higgs boson is predominantly created in the scattering process

$$\text{gluon} + \text{gluon} \rightarrow \text{Higgs Boson.}$$

The gluons, which have zero mass, collide head-on and in general have unequal energies.

(a) **[10 pts]** Suppose a Higgs boson of mass $m_H = 125\ \text{GeV}/c^2$ is produced in gluon-gluon fusion. The gluon moving in the $+z$ direction has energy $E_1 = 100$ GeV. What is the energy E_2 of the gluon moving in the $-z$ direction?

(b) **[10 pts]** Suppose that the Higgs boson from part (a) decays into two photons and that in the Higgs center-of-momentum frame, the photons are produced perpendicular to the beam axis. With what energies and what angles with respect to the beam axis are each of the photons produced in the lab frame?

Student Number:______________________________

5. A non-relativistic electron (charge $-e$ and mass m) is moving in the plane perpendicular to a uniform static magnetic field B. Its energy at the initial time $t = 0$ is $\mathcal{E}(0) = \mathcal{E}_0$.

   (a) [**8 pts**] Find the instantaneous power radiated (in the dipole approximation) at the time when the electron has energy $\mathcal{E}(t)$.

   (b) [**8 pts**] Find an expression for the energy of the electron as a function of time as it decreases due to the emission of dipole radiation.

   (c) [**4 pts**] After how many orbits does the electron energy decrease by a factor of 2, assuming a magnetic field of 1 Tesla?

Student Number:______________________________

PHYSICS 841 **Final Exam** **May 4, 2012**

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VECTOR CALCULUS

Gradient vector $= \vec{\nabla} = (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$
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Cylindrical coordinates (r, θ, z):

$$\nabla S = \frac{\partial S}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}_\theta + \frac{\partial S}{\partial z}\mathbf{e}_z$$

$$\vec{\nabla} \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$\begin{aligned}\vec{\nabla} \times \mathbf{v} \;=\; & \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right]\mathbf{e}_r \\ & + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\mathbf{e}_\theta \\ & + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\mathbf{e}_z\end{aligned}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

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$$(\mathbf{A}, \mathbf{B}) \Rightarrow \sqrt{\frac{4\pi}{\mu_0}}\,(\mathbf{A}, \mathbf{B}),$$

$$c = \sqrt{\frac{1}{\epsilon_0\mu_0}}.$$

Student Number:____________________

1. A spherical conductor of radius a and with total charge Q is placed concentrically inside a hollow spherical shell of radius R. The electric potential on the shell is given by the function $V(\theta) = V_0 \sin^2 \theta$, where $\{r, \theta, \phi\}$ are spherical coordinates.

   (a) [**12 pts**] Find the potential $\Phi(r, \theta, \phi)$ for $a \leq r \leq R$.

   (b) [**4 pts**] Find the electric field $\mathbf{E}(r, \theta, \phi)$ for $a \leq r \leq R$.

   (c) [**4 pts**] Find the surface charge density $\sigma(\theta, \phi)$ on the inner conducting sphere.

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2. Consider a very long coaxial cylindrical capacitor formed by two cylindrical shells with radii $r = a$ and $r = b > a$. The charge per unit length on the inner and outer cylindrical shells are $+\lambda$ and $-\lambda$ respectively. The inner cylindrical shell also carries a steady electric current I, uniformly spread over its surface and parallel to its symmetry axis.

   (a) [**4 pts**] Find the electric field $\mathbf{E}(\mathbf{r})$ in the space between the cylindrical shells.

   (b) [**4 pts**] Find the magnetic field $\mathbf{B}(\mathbf{r})$ in the space between the cylindrical shells.

   (c) [**4 pts**] Find the Poynting vector $\mathbf{S}(\mathbf{r})$ in the space between the cylindrical shells.

   (d) [**4 pts**] Find the total energy flux through a cross-sectional surface through the capacitor.

   (e) [**4 pts**] Is it possible to find an inertial frame where the electric field is zero inside the capacitor? If yes, give a relationship among the parameters of the capacitor that must be satisfied for this to be possible. If no, explain.

Student Number:____________________

3. A spherical shell of radius R is uniformly charged over its surface with a total charge Q. The sphere is rotating around an axis through its center with the angular speed ω. Let the center of the sphere be the origin, and let the rotation axis be the z axis, so that the angular velocity is $\boldsymbol{\omega}_1 = \omega\,\hat{\mathbf{z}}$.

   (a) [**8 pts**] Calculate the magnetic moment of the rotating spherical shell.

   (b) [**4 pts**] What is the magnetic field due to the rotating sphere at a point $\mathbf{r} = d\,\hat{\mathbf{x}}$, where $d \gg R$?

   (c) [**4 pts**] A second, identical charged spherical shell, is rotating with the same angular speed, but along the x axis, so that its angular velocity is $\boldsymbol{\omega}_2 = \omega\,\hat{\mathbf{x}}$. If the second shell is placed at the point $\mathbf{r}$ specified in (b), what is the torque (magnitude and direction) on the second spherical shell ?

   (d) [**4 pts**] What is the net force (magnitude and direction) on the spherical shell in part (c)?

Student Number:________________________

4. **[20 pts]** A neutral pion π^0 (mass 135 MeV/c^2) decays in flight into two photons of equal energy. The angle between the momenta of the photons is 60°. What was the velocity of the initial pion?

Student Number:____________________________

5. An antenna is made up of two identical linear antennae of length a, centered on the origin and oriented along the x and y axes. The two antennae carry AC current with the same amplitude and frequency, but with a relative phase α. We can represent the total current in the antenna by

$$\mathbf{I}(t) \ = \ \hat{\mathbf{x}}\, I_0 \cos(\omega t) + \hat{\mathbf{y}}\, I_0 \cos(\omega t + \alpha) \ .$$

Assume that the wavelength of the radiation is much larger than the length a.

(a) **[15 pts]** Find the angular distribution of the radiation averaged over one period of the oscillating current.

(b) **[5 pts]** Find the total intensity of the radiation averaged over one period of the oscillating current.

PHY-841: CLASSICAL ELECTRODYNAMICS I
FINAL EXAM; 100 points
May 3, 2010

NUMBER...

1. /20/ A compass needle, a magnetic dipole carrying the magnetic moment μ, can rotate in the horizontal xz-plane when applying a uniform static magnetic field $B = B_z$. The moment of inertia of the needle is equal to J. Find the period of small vibrations of the needle around its equilibrium position.

2. /20/ A small static magnetic moment $\vec{\mu}$ and a static electric charge e are placed at the origin.

a. Find the magnitude and direction of the Poynting vector of the electromagnetic field created by this system.

b. Calculate the energy flux through a spherical surface enclosing the origin as its center.

3. /20/ An electron (mass m, electric charge $-e$) is moving at a large distance in the Coulomb field of a nucleus of charge Ze (the impact parameter b is much greater than the radius of the nucleus R). The initial velocity of the electron far away from the nucleus is $v_0 \ll c$. Assume that the trajectory can be approximated by a straight line and the moment $t = 0$ corresponds to the closest approach (at a distance b) to the nucleus.

a. Calculate the loss of energy by dipole radiation over the whole trajectory of the electron.

b. Estimate the minimum impact parameter for the approximation of unperturbed straight line trajectory to be valid if $v_0 = 0.1c$ and $Z = 70$.

4. /20/ A neutral pion π° decays in flight into two photons. Find the minimum value of the opening angle between the momenta of the photons as a function of the velocity v of the pion.

5. **Quick questions** .

(a) /5/ Is it possible to create in a region of space an electrostatic field $\mathbf{E}(\mathbf{r}) = [\mathbf{a} \times \mathbf{r}]$, where $\mathbf{a}$ is a constant vector?

(b) /5/ A nucleus can be modeled by a uniformly charged axially symmetric ellipsoid. What are the non-zero components of the quadrupole tensor in the coordinate frame with the z-axis along the symmetry axis and the xy-plane in the equatorial cross section? Do the components of this tensor depend on the choice of the origin of coordinates and orientation of the axes?

(c) /5/ A plane capacitor is connected to an external circuit with the conductance current I. Find the magnitude of the displacement current when this capacitor is discharging.

(d) /5/ Two point charges e_1 and e_2 are at a distance a from each other, and the vector **a** is along the y-axis. This system is moving in the laboratory with the velocity $V = V_x$. What is the interaction force between the charges measured in the laboratory?

PHY-841: CLASSICAL ELECTRODYNAMICS - I

Subject Exam: Total = 100 points

January 14, 2010

NUMBER..................

1. /20/ Determine the electrostatic potential at large distances for a plane molecule of four point atoms located in the corners of a square of side a in the plane x, y: (1) $x = y = 0$, charge e; (2) $x = 0$, $y = a$, charge $-e$; (3) $x = a$, $y = 0$, charge $-e$; (4) $x = y = a$, charge e.

2. /20/ A light source emits light of frequency ω_0 with the wave vector $\mathbf{k}$ in the xy-plane, $k_x > 0$. The light is reflected from the plane mirror parallel to the yz-plane. The incidence angle (between the normal to the mirror and the wave vector of the incident wave) is α_0. Then the entire device (the source and the mirror) is brought into motion, with velocity v (with respect to the laboratory) along the x-axis. Predict the results of the measurements made in the laboratory for

a. frequencies of incident and reflected waves;

b. incidence angle;

c. relation between the incidence angle and the reflection angle (both in the lab frame).

3. /20/ A neutral atom contains a point-like nucleus of charge q and a spherically symmetric electron cloud with charge density $\rho_e(r)$.

a. Find the magnitude and the direction of electric field at an arbitrary point in terms of the function $\rho_e(r)$.

b. Consider the hydrogen atom in the ground state where $q = e$ and

$$\rho_e(r) = Ce^{-2r/a}, \tag{14}$$

$a = \hbar^2/m_e e^2$ is the Bohr radius. Calculate the constant factor C.

c. Calculate the electric field created by the hydrogen atom in the ground state.

d. Calculate the magnitude (in Volt/cm) of the electric field at $r = a$ and $r = 10a$.

4. /20/ A sphere of radius R is uniformly charged over its volume with the total charge e. The sphere is rotating with the angular velocity $\vec{\Omega}$ around an axis through the center of the sphere.

a. Calculate the magnetic moment of the rotating sphere.

b. Consider the sphere as a model of the electron with the *classical electron radius* $R = r_e \equiv e^2/m_e c^2$ where m_e is the electron mass. Calculate the angular frequency which is needed in order to explain the intrinsic magnetic moment of the electron equal to one *Bohr magneton* $e\hbar/2m_e c$. Give a number for Ω and express your opinion about validity of the model.

5. /20/ A plane monochromatic electromagnetic wave of frequency ω propagates along the x-direction. The wave is linearly polarized along the y-direction. The wave is being received by a frame antenna which is shaped as a square of size a in the xy-plane. Calculate the signal (e.m.f.) induced in the antenna. What size a corresponds to the maximum magnitude of the signal?

PHY 841 Final Exam

Read all of the following information before starting the exam:

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- Show all work (neatly as possible and in logical order) to maximize your credit. Circle or otherwise indicate your final answers.
- All work should be shown in the space after each question. If you need extra space, use the blank pages that are attached and indicate clearly what problem it corresponds to. You are not allowed to use your own scrap paper.
- This test has 6 problems. Please make sure that you have all of the pages.
- Good luck!

SELECTED NUMERICAL DATA

Speed of light $c = 3 \times 10^{10}\,\text{cm/s}$,
Elementary charge $e = 4.8 \times 10^{-10}$ abs. units,
Planck constant $\hbar = h/2\pi = 1.055 \times 10^{-27}\,\text{erg}\cdot\text{s} = 6.582 \times 10^{-22}\,\text{MeV}\cdot\text{s}$.
Do not use these numbers directly! Instead combine your expressions in the standard combinations:
Fine structure constant (dimensionless) $\alpha = e^2/\hbar c$, $1/\alpha = 137.036$;
$\hbar c = 197.3\,\text{MeV}\cdot\text{fm} \approx 2 \times 10^{-5}\,\text{eV}\cdot\,\text{cm}$ ($1\,\text{fm} = 10^{-13}\,\text{cm}$).

Electron mass $m = 0.911 \times 10^{-27}\,\text{g}$=$0.511\,\text{MeV}/c^2$,
Proton mass $m_p = 1.673 \times 10^{-24}\,\text{g}$=$938.3\,\text{MeV}/c^2 = 1836.2\,m$,

Compton wave length of the electron $\lambda_e = \hbar/mc = 3.862 \times 10^{-11}\,\text{cm}$,
Classical electron radius $r_e = e^2/mc^2 = 2.818 \times 10^{-13}\,\text{cm}$.

Bohr magneton $\mu_0 = e\hbar/2mc = 9.274 \times 10^{-21}\,\text{erg/Gs}$,
Nuclear magneton (n.m.) $\mu_N = e\hbar/2m_pc = \mu_0(m/m_p) = 5.051 \times 10^{-24}\,\text{erg/Gs}$,
Proton magnetic moment $\mu_p = 2.793\,\text{n.m.}$,
Neutron magnetic moment $\mu_n = -1.913\,\text{n.m.}$

Gravitational constant $G = 6.67 \times 10^{-8}\,\text{cm}^3\text{g}^{-1}\text{s}^{-2}$.

Useful to remember:

$$1\,\text{J} = 10^7\,\text{erg},\ 1\,\text{eV}/c^2 = 1.783 \times 10^{-33}\text{g}$$

$$1\,\text{Coulomb} = 3 \times 10^9\,\text{abs.units}, 1\,\text{Tesla} = 10^4\,\text{Gs}$$

$$1\,\text{Volt} = \frac{1}{300}\,\text{abs.units}$$

$$1\,\text{Ampere} = 3 \times 10^9\,\text{abs.units}$$

$$1\,\text{Watt} = 10^7\,\text{erg/s}$$

Conversion from Gaussian to SI units:

$$(\rho, \mathbf{j}) \Rightarrow \frac{(\rho, \mathbf{j})}{\sqrt{4\pi\epsilon_0}},$$

$$(\phi, \mathbf{E}) \Rightarrow \sqrt{4\pi\epsilon_0}\,(\phi, \mathbf{E}),$$

$$(\mathbf{A}, \mathbf{B}) \Rightarrow \sqrt{\frac{4\pi}{\mu_0}}\,(\mathbf{A}, \mathbf{B}),$$

$$c = \sqrt{\frac{1}{\epsilon_0\mu_0}}.$$

Spherical Harmonics

$$l = 0: \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1: \qquad Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

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$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

Legendre Polynomials

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l,0}(\theta,\phi)$$

VECTOR CALCULUS

Gradient vector $= \vec{\nabla} = (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$;
$\operatorname{div}\mathbf{v} \equiv (\vec{\nabla}\cdot\mathbf{v})$
$\operatorname{curl}\mathbf{v} \equiv [\vec{\nabla}\times\mathbf{v}]$.

Cylindrical coordinates (z, r, α):

$$\nabla S = \frac{\partial S}{\partial z}\mathbf{e}^{(z)} + \frac{\partial S}{\partial r}\mathbf{e}^{(r)} + \frac{1}{r}\frac{\partial S}{\partial\alpha}\mathbf{e}^{(\alpha)};$$

$$\operatorname{div}\mathbf{v} = \frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\alpha}{\partial\alpha};$$

$$\begin{aligned}\operatorname{curl}\mathbf{v} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\alpha) - \frac{\partial v_r}{\partial\alpha}\right]\mathbf{e}^{(z)} \\ &+ \left[\frac{1}{r}\frac{\partial v_z}{\partial\alpha} - \frac{\partial v_\alpha}{\partial z}\right]\mathbf{e}^{(r)} \\ &+ \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\mathbf{e}^{(\alpha)};\end{aligned}$$

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\alpha^2}.$$

Spherical polar coordinates (r, θ, α):

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$$\operatorname{div}\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\alpha}{\partial\alpha};$$

$$\begin{aligned}\operatorname{curl}\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta v_\alpha) - \frac{\partial v_\theta}{\partial\alpha}\right]\mathbf{e}^{(r)} \\ &+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\alpha} - \frac{\partial}{\partial r}(r v_\alpha)\right]\mathbf{e}^{(\theta)} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\mathbf{e}^{(\alpha)};\end{aligned}$$

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1. (*15 points*) Two spheres of radii R_1 and R_2 are charged with densities $\rho_1(r_1)$ and $\rho_2(r_2)$ which are spherically symmetric (r_1 and r_2 are distances from the corresponding centers). The distance between the centers is $a > R_1 + R_2$. Derive Gauss's famous result that the interaction energy for such spherically symmetric charge distributions is the same as if the spheres were replaced by two point charges q_1 and q_2 with a separation of a.

2. (*15 points*) The total charge q is uniformly distributed over the volume of a solid cylinder of radius R and height h. The cylinder is rotating around its geometrical axis with the angular velocity Ω. An identical cylinder is placed parallel to the first one at a distance d between the centers ($d \gg R, h$), and is rotating around its own axis with the same angular velocity in the opposite direction.

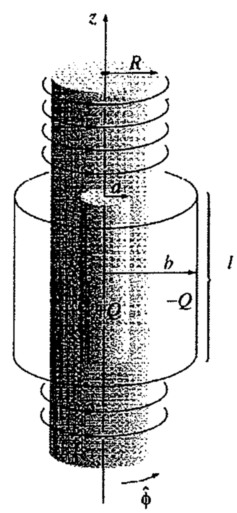
a. (*8 pts*) Determine the magnitude and directions of the magnetic moments of the cylinders.

b. (*7 pts*) Find the interaction energy and the force of the magnetic interaction between the cylinders and determine if the cylinders repel or attract each other.

3. (*15 points*) The scattering of a fast proton off a fixed hydrogen target is generally accompanied by the production of multiple pions,

$$p + p \Rightarrow p + p + N\pi. \tag{1}$$

The momentum of the fast proton is $p_1 = 5\,\mathrm{GeV}/c$ (1GeV=10^3 MeV). Evaluate the maximum possible number N of created pions; assume $m_p = 938$ MeV and $m_\pi = 140$ MeV neglecting the difference in masses of charged and neutral pions.



4. (*20 points*) Consider a very long solenoid with radius R, current I, and n turns per unit length. Coaxial with the solenoid are two long cylindrical shells of length l. One cylinder of radius a is inside the solenoid and carries a uniform charge $+Q$ on its surface. The other cylinder of radius b is outside the solenoid and carries a uniform charge of $-Q$ on its surface. Now imagine the current is slowly reduced to zero. The resulting induced **E**-field will set the cylinders into rotation.

a. (*8 pts*) What is the angular momentum of the system before we start to turn the current off? What direction will the two cylinders start to rotate?
b. (*12 pts*) Calculate the torques and the corresponding final angular momentum for each cylinder after the current has been extinguished. Is angular momentum conserved? You may ignore the magnetic fields generated by the convection currents of the rotating cylinders.

5. (*15 points*) In Neils Bohr's early quantum theory of hydrogen, the electron in its ground state is supposed to travel in a circle of radius $5 \times 10^{-11}\, m$, held in orbit by the Coulomb attraction to the proton nucleus. According to classical EM, this electron should radiate, and thus spiral in to the nucleus. Show that the condition $v \ll c$ would be satisfied for most of the journey, and use the non-relativistic Larmor formula to calculate the lifetime of the Bohr atom. You may assume each revolution is circular.

6. (*20 points*) Quick questions.

a. (*5 pts*) Consider a cube of length L on each side with charges $+q$ glued to each corner. Naively, one might think that a free charge q deposited in the center would be suspended in midair since it would be repelled equally from each corner. In two or three sentences, explain why this arrangement is not a stable equilibrium. This is the essence of Earnshaw's theorem, which states that a charged particle cannot be held in a stable equilibrium by electrostatic forces alone.

b. (*5 pts*) A plane capacitor with a narrow spacing between the plates is connected to an external circuit with the conductance current I. Find the magnitude of the displacement current when this capacitor is discharging.

c. (*5 pts*) Take a short cylindrical iron magnet (which can be idealized as a solenoid) and drop it down a vertical copper pipe of a larger diameter. It takes several seconds for it to emerge at the bottom, whereas an otherwise identical piece of unmagnetized iron makes the trip in a fraction of a second. Explain this behavior with a picture and a few sentences.

d. (*5 pts*) Consider a reference frame K where $\mathbf{E} \cdot \mathbf{B} = 0$ and $|\mathbf{E}| > |\mathbf{B}|$. Is it possible to find a frame of reference where the electric field vanishes?

Secret Student Number:

PHY 841 Final Exam

1/10/2012

Read all of the following information before starting the exam:

- All problems are in Gaussian units. A list of common physical constants and parameters is provided. You may use a simple calculator, but no external notes, books, etc.
- Show all work (neatly as possible and in logical order) to maximize your credit. Circle or otherwise indicate your final answers.
- All work should be shown in the space after each question. If you need extra space, use the blank pages that are attached. You are not allowed to use your own scrap paper.
- This test has 6 problems. Please make sure that you have all of the pages.
- Good luck!

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1. (*15 points*) Consider two spherical shells. The inner shell of radius a is at constant potential V_a, while the outer shell of radius b is at constant potential V_b. Find the electric potential in all regions of space.

2. (*15 points*) Consider a cylinder of radius R and height h that has a uniform charge density ρ. The cylinder is immersed in a uniform time-dependent external magnetic field $B = B_0 e^{-t/\tau}$ that points in the z-direction.

a. (*7 pts*) Find the induced electric field (magnitude and direction) at a distance r from the z-axis.

b. (*8 pts*) Find the torque $\mathbf{N} = \int \mathbf{x} \times d\mathbf{F}$ on the cylinder and the final angular momentum (recall $d\mathbf{L}/dt = \mathbf{N}$) after the external magnetic field has decayed to zero at $t \to \infty$.

3. (*15 points*) A neutral pion π° (mass 135 MeV) decays in flight into two photons of equal energy. What was the velocity of the initial pion in terms of the angle between the emitted photons?

4. (*15 points*) A non-relativistic proton is moving in the plane perpendicular to a uniform static magnetic field **B**. Its energy at the initial moment is equal to $\mathcal{E}(t = 0) = \mathcal{E}_0$. Find how this energy is decreases with time due to the dipole radiation, and determine after how many turns the proton energy decreases by a factor of 2 assuming a magnetic field of 10^4 Gauss (note: 1 Tesla = 10^4 Gauss).

5. (*20 points*) EM Waves and Radiation pressure.

a. (*10 pts*) Consider a monochromatic plane wave traveling in the z-direction incident on a screen in the xy-plane. The electric field for such a plane wave is $\mathbf{E} = E_0\hat{\mathbf{x}}\cos(kz - \omega t)$. If the screen is a perfect absorber of the light, what is the (time-averaged) radiation pressure on the screen?

b. (*10 pts*) Imagine the screen is made out of material that has a uniform conductivity σ (recall Ohm's Law $\mathbf{J} = \sigma\mathbf{E}$). By considering the directions of $\mathbf{E}$ and $\mathbf{B}$, as well as the current density $\mathbf{J}$ in the screen, explain with qualitative arguments how the time-averaged radiation pressure can be understood as arising solely from the magnetic force on the mobile charges in the screen.

6. (*20 points*) Quick questions.

a. (*4 pts*) Write the Maxwell's equations in differential and integral form.

b. (*4 pts*) Prove that the continuity equation (the differential equation that expresses the conservation of charge) is contained in Maxwell's equations.

c. (*4 pts*) For electrostatics, prove that the charge density in the interior of a conductor must vanish.

d. (*4 pts*) An infinite wire carries a steady current I and runs along the z-axis. Find the magnetic field $\mathbf{B}$ due to the wire.

e. (*4 pts*) Consider static and uniform fields $\mathbf{E} = E\hat{\mathbf{y}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$. A point charge q is released at the origin. Without solving equations of motion, sketch the motion of the point charge and give a qualitative explanation justifying your sketch.