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MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

Subject Exam

June 1, 2021

Student Number:

Points					
F1	F2	F3	F4	F5	total
10	10	10	10	10	50

- **Read through the exam once before starting to work. Not all questions are equally difficult, and you may wish to start with problems that play to your strengths.**
- **Document all your work (including scratch paper!) – this helps with assigning partial credit.**
- **Justify all your answers!**
- **The formula sheet should contain everything you will need.**
- **Most importantly, do not hesitate to ask if anything is unclear!**

Good Luck!

Problem F1 – Constraints in a Funnel

[10 Points] A bead of mass m is rolling on the inner mantle of a cone-shaped funnel. For simpliciy, we model the mantle as $z(\rho) = \alpha\rho$ (with $\alpha > 0$) in cylindrical coordinates ρ, ϕ, z (see side view shown in figure).

1. Construct the **unconstrained** Lagrangian L in cylindrical coordinates. (You can treat the bead as a point mass.) Write the constraint in the form $f(\rho, \phi, z) = 0$, and couple it to L by introducing a Lagrange multiplier λ.

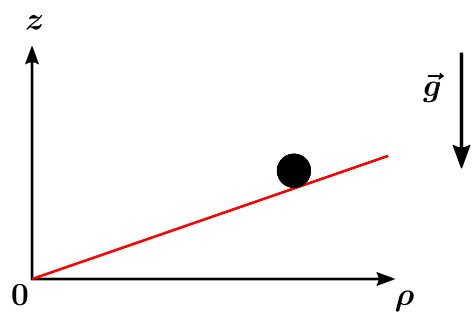
2. Find at least two conserved quantities and either argue or demonstrate explicitly that they are conserved.

3. Determine the complete set of Lagrange equations for ρ, ϕ, z and λ.

4. Using the Lagrange equations, show that

$$|\lambda| = \frac{1}{1+\alpha^2}\left(\alpha\frac{l_z^2}{m\rho^3} + mg\right) , \qquad (1)$$

   where l_z is the z component of the angular momentum. What is the *physical* interpretation of this constraint force?



Problem F2 – A Central Force

[10 Points] A particle of mass m is moving in the central force field

$$\vec{F}(\vec{r}) = \left(-\frac{\alpha}{r^2} - \frac{\beta}{r^3}\right)\vec{e}_r\,, \quad \alpha, \beta > 0\,. \tag{2}$$

1. Compute the potential $V(r)$, choosing any integration constants such that the potential vanishes for $r \to \infty$. Sketch $V(r)$ and indicate which term dominates at short and long distances, respectively.

2. Now consider the effective potential. How must $V_{\text{eff}}(r)$ behave at short distances to support stable orbits, i.e., orbit that do not reach the origin or infinity? What is the critical angular momentum l_c that an object must have to move in a stable orbit? Sketch $V_{\text{eff}}(r)$ for $l < l_c$ and $l > l_c$.

3. For $l > l_c$, determine the radius and energy of *circular* orbits.

4. For what *range* of energies E will an orbit with $l > l_c$ be bound? Determine the turning points r_{min} and r_{max} of such bound orbits as a function of α, β, E and l.

Problem F3 – Normal Modes

[10 Points] Consider two identical masses m that can move on a circular horizontal track of radius R (see figure). Each of the masses is connected to a fixed point by identical springs with constant $2k$, and a spring with constant k connects the masses to each other.

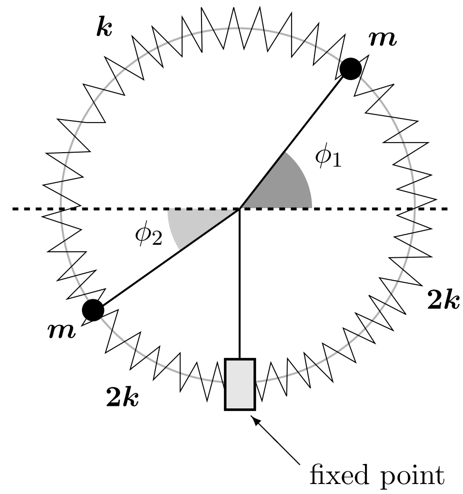
1. Construct the Lagrangian for the system in terms of the (counterclockwise) angular displacements ϕ_1 and ϕ_2 of the two masses from their equilibrium positions, as shown in the diagram.

   HINT: Distances on the circular track can be expressed in terms of arc lengths.

2. Determine the normal modes, i.e., characteristic frequencies and vectors (vectors do not need to be normalized). Sketch and interpret your solutions.

3. Now the fixed point is released, so that the system can rotate freely on the circular track. The track itself remains at rest. How will the characteristic frequencies change *qualitatively* as a result?

   HINT: A calculation is not necessary, but if you explicitly want to check, you can use that two springs connected "in series" can be replaced by a single spring with constant $k_{\text{eff}} = (\frac{1}{k_1} + \frac{1}{k_2})^{-1}$.



Problem F4 – Rotations of a Solid Disk

[10 Points] A thin homogeneous disk of mass M and radius R in the xy-plane is described by the mass distribution

$$\rho_M(\vec{r}) = C\,\theta(R-\rho)\delta(z). \tag{3}$$

1. Compute the volume integral over $\rho_M(\vec{r})$ to show that $C = M/(\pi R^2)$.
2. Compute the disk's moment of inertia tensor for rotations with respect to the center of mass.

   HINT: You can use symmetries and the properties of $\rho_M(\vec{r})$ to argue that only two diagonal matrix elements of $\hat{\boldsymbol{I}}$ need to be computed explicitly. If you do, make sure to explain your reasoning!
3. Use the parallel axis theorem to compute the moment of inertia tensors with respect to the points $\vec{R} = R\vec{e}_z$ and $\vec{R} = R\vec{e}_x$, where the unit vectors refer to the originally chosen coordinate system.

Problem F5 – A Simple Toda Chain

[10 Points] Consider a system of three identical particles of mass m whose dynamics is described by the Lagrangian

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V_0 e^{a(x_3 - x_2)} - V_0 e^{a(x_2 - x_1)} \,. \tag{4}$$

1. Determine the canonical momenta p_1, p_2, p_3, and use them to construct the Hamiltonian for the system.

2. Derive Hamilton's equations.

3. Show that the Poisson bracket $\{p_1 + p_2 + p_3, H\}$ vanishes, which means that $p_1 + p_2 + p_3$ is a conserved quantity. What is its physical interpration, and why is it conserved in this case?

4. Find (at least) one additional conserved quantity, and either argue or demonstrate its conservation through an explicit calculation.