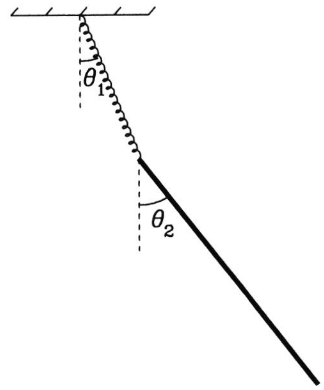
Classical Mechanics Subject Exam   August 29, 2016   NAME ______________________________

1. [10 pts] Consider the Lagrangian $\mathcal{L} = \frac{1}{2} M\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + K\,x\,\dot{y}$ where x and y are generalized coordinates and K and M are constants.

   Change variables in this Lagrangian to cylindrical coordinates $x = r\cos\phi$, $y = r\sin\phi$, $z = z$. Then find the *Hamiltonian* as a function of the new coordinates r, ϕ, z and their appropriate canonical momenta.

2. [10 pts] A uniform thin rod of length b and mass M hangs from the ceiling by a massless spring that has spring constant k and unstretched length r_0.



(a) Calculate the potential energy of the system (due to gravity and the spring) as a function of θ_1, θ_2, and r = length of the spring.

(b) Calculate the kinetic energy of the system as a function of θ_1, $\dot{\theta}_1$, θ_2, $\dot{\theta}_2$, r, and $\dot{r}$. (Hint: The moment of inertia of the rod about its center of mass is $Mb^2/12$.)

3. [10 pts] Find the normal mode oscillation frequencies for a system whose coordinates x and y obey

$$\begin{aligned} \ddot{x} + \ddot{y} &= -2\,x \\ \ddot{x} + 2\,\ddot{y} &= -3\,y \end{aligned}$$

4. [10 pts] A hockey puck is approximately a uniform cylinder with a thickness of 1 inch and a diameter of 3 inches. (1 inch = 2.54 cm — Canadians had not yet gone metric when they invented the game!) The mass of the puck is 0.16 kg. Calculate its principal moments of inertia.

   Let M = mass of the puck, H = its height, and D = its diameter. You do not need to put the actual numerical values into your final answer.

5. [10 pts] The moment of inertia tensor of a rigid body, in the frame in which it is diagonal, is given by

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Find the moment of inertia tensor of this body in the coordinate frame that is obtained by rotating it about the z axis through an angle of $30°$.

6. [10 pts] With a convenient choice of units, the Lagrangian for a particular continuous system can be written as

$$L = \frac{1}{2}\int_0^1 \left[y^2 + x\left(\frac{\partial y}{\partial t}\right)^2 - x^2\left(\frac{\partial y}{\partial x}\right)^2 \right] dx$$

(a) Find the equation of motion, which is a partial differential equation for $y(x,t)$.

(b) Assume an oscillating solution to the equation of motion and find the resulting ordinary differential equation for the displacement function $A(x)$.