Statistical Mechanics Subject Exam, Monday, Dec. 11, 2017

SECRET STUDENT NUMBER: STUDNUMBER

DO NOT WRITE YOUR NAME OR STUDENT NUMBER ON ANY SHEET!

FUN FACTS TO KNOW AND TELL

$$\begin{aligned}
\int_0^\infty dx\, \frac{x^{n-1}}{e^x - 1} &= \Gamma(n)\zeta(n), \qquad \int_0^\infty dx\, \frac{x^{n-1}}{e^x + 1} = \Gamma(n)\zeta(n)\left[1 - (1/2)^{n-1}\right], \\
\zeta(n) &\equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!, \\
\zeta(3/2) &= 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90},
\end{aligned}$$

$$\int_{-\infty}^{\infty} dx\; e^{-x^2/2} = \sqrt{2\pi}, \qquad \int_0^\infty dx\; x^n e^{-x} = n!$$

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LONG ANSWER SECTION

1. (10 pts) Beginning with:

$$dE = TdS - PdV + \mu dN,$$

derive the Maxwell relation,

$$\left.\frac{\partial V}{\partial \mu}\right|_{S,P} = -\left.\frac{\partial N}{\partial P}\right|_{S,\mu}.$$

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Extra workspace for #1

2. Consider the equation of state,

$$P = \rho T e^{\rho/\rho_0} - a\rho^2/\rho_0,$$

where ρ is the number density, T is the temperature, and ρ_0 and a are both positive constants.

(a) (10 pts) Find the critical density and critical temperature related to the first-order phase transition.

(b) (5 pts) What is the liquid density as $T \to 0$?

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Extra workspace for #2

3. Consider a two-dimensional world (atoms move ONLY in the xy plane), where there is a two-dimensional metal where the longitudinal and transverse speeds of sound are both c_s. The material has spin 1/2 electrons of mass m_e and the density of electrons per area is ρ_e. The system is at a low temperature T. The temperature is much less than the Debye temperature and much less than the Fermi energy.

   (a) (10 pts) Find the contribution to the specific heat per area, $C_v^{(\mathrm{s})} = (1/A)dE/dT$, due to phonons.

   (b) (10 pts) Find the contribution to the specific heat per area, $C_V^{(\mathrm{f})}$, due to the Fermi motion of the electrons.

   Answers should be expressed in terms of T, c_s, m_e and ρ_e.

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Extra work space for #3

4. (10 pts) The density-density correlation function in a one-dimensional liquid of length L has the form

$$\langle \delta\rho(x=0)\delta\rho(x)\rangle = A\delta(x) + Be^{-|x|/\ell},$$

where A, B and ℓ are positive constants. The liquid has an average particle number density (number per unit length) $\rho_0 = N/L$
Find the charge fluctuation per particle, $\chi_{QQ} \equiv \langle \delta Q^2\rangle/N$. Give answer in terms of A, B, ℓ and ρ_0.

Extra work space for #4

5. (10 pts) A particle of mass m is thermally equilibrated at temperature T in a one-dimensional potential

$$V(x) = \begin{cases} V_0 \ln(x) + kx, & x > 0 \\ \infty, & x < 0 \end{cases}$$

Find $\langle x \rangle$ in terms of T, k and V_0.

Extra work space for #5

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SHORT ANSWER SECTION

6. (4 pts) Two phase transitions of the same universality class have: Circle all that are true
   (a) The same microscopic degrees of freedom
   (b) The same critical exponents
   (c) The same critical temperature
   (d) The same dimensionality
   (e) The same symmetry breaking

7. (4 pts) Two species of ink molecules diffuse through a liquid. Species A has twice the mass as species B, $M_A = 2M_B$. They have identical the collision (relaxation) times, $\tau_{\text{coll},A} = \tau_{\text{coll},B}$. What is the ratio of the diffusion constants, D_A/D_B?

   ____________________.

8. (2 pts each) Two identical spin-one bosons can each occupy one of two single-particle energy levels, 0 and ϵ.
   (a) What is the average energy when $T = 0$? ____________
   (b) What is the entropy when $T = 0$? ____________
   (c) What is the average energy when $T >> \epsilon$? ____________
   (d) What is the entropy when $T >> \epsilon$? ____________

9. (4 pts) If you read an article where the authors minimize the Gibb's Free Energy to solve for an order parameter ϕ, which quantities can you assume were fixed as ϕ was varied? Circle all that are true.
   (a) entropy
   (b) temperature
   (c) particle number
   (d) density
   (e) chemical potential
   (f) pressure
   (g) energy density

10. (4 pts) Consider massless bosons existing in a system of dimension D. For what values of D would Bose condensation be possible?

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11. (4 pts) A non-relativistic Fermi gas exists in dimension D at zero temperature. The density ρ_D (number per hypervolume of dimension D) behaves as

$$\rho_D \sim \epsilon_F^\ell.$$

What is ℓ?

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12. (4 pts) A two-dimensional square lattice supports three kinds of sound waves: longitudinal, transverse in-plane and transverse out-of-plane. What is the specific heat per ion, $(1/N)dE/dT$, at high temperature?

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13. (3 pts) A low-density gas of O_2 molecules is thermalized in a container of fixed volume V and temperature $T \approx 200$ K. If there are N molecules, the average thermal energy of the gas is:

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