Student Number: 2

Statistical Mechanics Subject Exam

August 23rd, 2019

Do not write your name on the exam.

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \text{ (for } a > 0)$$

$$\Gamma(n) = (n-1)! = \int_0^{\infty} dx x^{n-1} e^{-x}$$

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m} = \frac{1}{\Gamma(m)} \int_0^{\infty} dx \frac{x^{m-1}}{e^x - 1}$$

$$\zeta(2) = \frac{\pi^2}{6} \quad \zeta(3) = \frac{\pi^4}{90} \quad \zeta(4) = \frac{\pi^6}{945}$$

$$\int_0^{\infty} \frac{x^{\nu-1} dx}{e^x + 1} = \Gamma(\nu)\zeta(\nu)[1 - 2^{1-\nu}]$$

$$\ln N! \approx N \ln N - N \text{ (for } N \gg 1)$$

1. Consider a three-dimensional classical system of N distinguishable, ultra-relativistic particles with energies $\epsilon = |\vec{p}|c + U(\vec{r})$ confined to a sphere with radius R. The potential is given by

$$U(\vec{r}) = \begin{cases} -U_0 & |\vec{r}| < \ell \\ 0 & |\vec{r}| > \ell \text{ and } |\vec{r}| < R \\ \infty & |\vec{r}| > R \end{cases} \tag{1}$$

where ℓ is a constant length scale that can be assumed to be much less than R.

   (a) (10 pts) Calculate the canonical partition function for this system.

   (b) (5 pts) Calculate the pressure of the system.

   (c) (5 pts) What is the pressure when $T \ll U_0$? Briefly describe what is happening physically in this limit compared to an ideal gas.

Extra room for problem 1.

2. Consider a system of N distinguishable particles where each of the particles is in one of four single-particle states denoted 1, 2, 3 and 4. The total number of particles in state i is denoted N_i and the fraction of particles in state i is denoted p_i.

   (a) (10 pts) Find the Boltzmann entropy of this system in the large N limit in terms of p_i.

   (b) (10 pts) Find the values of p_1, p_2, p_3, and p_4 that maximize the entropy.

Extra room for problem 2.

3. Consider a two dimensional lattice of N coupled oscillators with area A that can move in the x, y, and z directions. Assume both longitudinal and transverse sound waves have dispersion relation $\omega = c_s k$ where ω is the wave frequency, k is the wave number, and c_s is the sound speed for all waves.

   (a) (5 pts) Calculate the Debye frequency, ω_D for this system.

   (b) (5 pts) What is the specific heat of the system for $T \gg \hbar\omega_D$?

   (c) (10 pts) What is the specific heat for $T \ll \hbar\omega_D$?

Extra room for problem 3.

4. (5 pts) Prove the relation

$$\left(\frac{\partial P}{\partial T}\right)_{S,N} = \left(\frac{\partial S}{\partial V}\right)_{P,N}.$$

Extra room for problem 4.

5. Consider a gas of N electrons confined to two-dimensional area A with mass m in contact with a reservoir with temperature T and chemical potential μ.

   (a) (5 pts) Find the Fermi energy, ϵ_F of the system.

   (b) (5 pts) Calculate the pressure of the system when $T = 0$.

   (c) (10 pts) What is the specific heat of the electrons at fixed μ when $T \ll \epsilon_F$, to first order in T?

Extra room for problem 5.

Short Answer Section

6. (4 pts) Consider a system described by the Landau-Ginzburg Hamiltonian

$$H = \int_{-\infty}^{\infty} dx \left[\frac{t}{2} m^2 + u m^4 + \frac{\kappa}{2} \left(\nabla m \right)^2 \right] \tag{2}$$

where m is a scalar order parameter. Circle the following statements that are true:

(a) This Hamiltonian possesses a discrete symmetry in m.

(b) This Hamiltonian possesses a continuous symmetry in m.

(c) When this system is taken below it's critical temperature, domain walls will appear.

(d) When this system is taken below it's critical temperature, Goldstone modes will appear.

7. Assume that we have a system with an entropy given by $S(E, V, N, x)$, where x is an unconstrained parameter of the system that can adjust itself to maximize the entropy. S is the entropy, F is the Helmholtz free energy, G is the Gibbs free energy, Ω is the grand potential, and H is the enthalpy. For each, choose one of (minimizes/maximizes) and one of (S, F, G, H, Ω, H). (1 pt each)

(a) If the system equilibrates at fixed temperature, pressure, and number, x will take on a value that minimizes/maximizes the potential S, F, G, Ω, H.

(b) If the system equilibrates at fixed energy, pressure, and number, x will take on a value that minimizes/maximizes the potential S, F, G, Ω, H.

(c) If the system equilibrates at fixed temperature, volume, and chemical potential, x will take on a value that minimizes/maximizes the potential S, F, G, Ω, H.

8. Consider a system containing two indistinguishable spin-1/2 Fermions with three spin-independent single-particle energy levels $-\epsilon$, 0, and ϵ that is thermalized at temperature T. (2 pts each)

(a) What is the average total energy when $T = 0$?

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(b) What is the entropy when $T = 0$?

(c) What is the average total energy when $T \to \infty$?

(d) What is the entropy when $T \to \infty$?