Secret student number:

QM Subject Exam, Fall 2015

Read all of the following information before starting the exam:

- Make sure your secret student number is written at the top of every page. Do not write your name on the exam.
- Show all work (neatly as possible and in logical order) to maximize your credit. Circle or otherwise indicate your final answers.
- All work should be shown in the space after each question. If you need extra space, use the blank pages that are attached and indicate clearly what problem it corresponds to. You are not allowed to use your own scrap paper.
- This test has 5 problems for a total of 100 points. Please make sure that you have all of the pages.
- Good luck!

1. (*20 points*) The electron neutrino and tau neutrino mixing in matter is described by the Hamiltonian (in units where $\hbar = c = 1$),

$$H = -\omega\Big(\cos(2\theta)\sigma_3 - \sin(2\theta)\sigma_1\Big) + \frac{GN}{\sqrt{2}}\sigma_3$$

where

- $\sigma_{1,3}$ are Pauli matrices
- $\omega = (m_{\nu_\tau}^2 - m_{\nu_e}^2)/(2E)$
- θ is the vacuum mixing angle
- G is the Fermi decay constant
- N is the electron density in matter
- E is the relativistic neutrino energy

a. (*10 pts*) Find the eigenvalues of H. (Write $H = \alpha\sigma_3 + \beta\sigma_1$ to simplify algebra.)

b. (*10 pts*) Parameterize the eigenvectors of H as linear combinations of the free electron $|\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and free tau neutrino $|\nu_\tau\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ states, i.e.,

$$|\nu_+\rangle = +\cos\theta_m |\nu_e\rangle + \sin\theta_m |\nu_\tau\rangle$$
$$|\nu_-\rangle = -\sin\theta_m |\nu_e\rangle + \cos\theta_m |\nu_\tau\rangle$$

where θ_m is the mixing angle in matter whose explicit form depends on θ, ω, G, N. (you are NOT required to derive the explicit form). An electron neutrino enters a region of matter with uniform N at $t = 0$. What is the probability for it to turn into a tau neutrino after propagating thru the matter for a time t?

2. (*20 points*) Short questions.
a. (*5 pts*) Let $|(ls)jm\rangle$ denote the angular momentum state of an electron that results from coupling its orbital (l) and intrinsic ($s = 1/2$) angular momentum to a total angular momentum j and projection m. Consider $|\Phi\rangle = \vec{\sigma} \cdot \vec{r}|(3\frac{1}{2})\frac{7}{2}\frac{7}{2}\rangle$. Using symmetry arguments, find i) what j, m components does $|\Phi\rangle$ contain, ii) what is the parity of $|\Phi\rangle$, and iii) express $|\Phi\rangle$ in terms of the appropriate $|(l\frac{1}{2})jm\rangle$ (don't worry about evaluating numerical factors.)

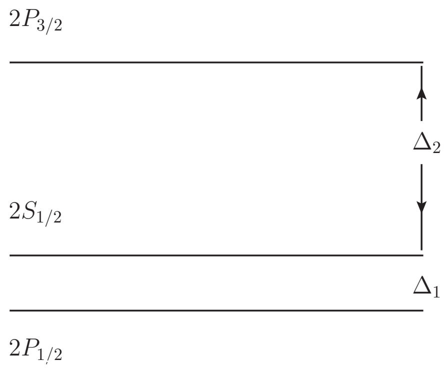
b. (*5 pts*) Consider matrix elements of a rank-1 spherical tensor operator between an initial state $|\psi_i\rangle$ comprised of a spin 1/2 particle and a spin 3/2 particle, and some final state $\langle\psi_f|$. What values of total angular momentum j for ψ_f are possible for the matrix elements to be non-vanishing?

c. (*5 pts*) Two identical spin 1/2 fermions interact via a central potential $V(|\vec{r}|)$ to form a bound state. What are the allowable l, S values (l is the relative orbital angular momentum, S the total spin) for the bound state?

d. (*5 pts*) Prove the eigenvalues of a Hermitian operator ($A = A^\dagger$) are real.

3. (*20 points*) A charged particle is in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, and is subjected to a harmonic oscillator potential in the z-coordinate. Find the energy spectrum and corresponding energy eigenfunctions.

4. (*20 points*)
Consider the Stark effect ($V = -e\mathcal{E}z$) in the $n = 2$ states of Hydrogen. The unperturbed energy levels, which include the fine structure and Lamb shift splittings, are as shown in the figure. You may assume throughout that any mixing with the $n \neq 2$ states is completely negligible. **In the following, you don't actually have to evaluate the matrix elements of V.**



a. (*15 pts*) Using symmetry, show that the perturbation, which is naively an 8x8 matrix in the $n = 2$ multiplet of states, splits into block diagonal form. Indicate the structure of each block (i.e., the size of the block, the states $|nljm\rangle$ involved, and which elements are non-zero). Indicate if any of the non-vanishing matrix elements are equal, and if so, explain why.

b. (*5 pts*) In the limit of $|e\mathcal{E}a_0| \ll \Delta_1$ where a_0 is the Bohr radius, indicate how you would calculate the first non-vanishing energy shifts in perturbation theory.

5. (*20 points*) Consider an electron in 1D that is bound in a harmonic oscillator potential and is subjected to a weak time-dependent electric field $E(t) = E_0 e^{-|t|/\tau}$. At $t = -\infty$, the electron is in the harmonic oscillator ground state. Find the probability that the electron is excited to the n^{th} excited state of the oscillator potential at $t = \infty$. You might find it useful to recall that $x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$.

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