Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi=0$$

$$\nabla\cdot(\nabla\times\mathbf{a})=0$$

$$\nabla\times(\nabla\times\mathbf{a})=\nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a})=\mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a})=\nabla\psi\times\mathbf{a}+\psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r=|\mathbf{x}|$, and $\mathbf{n}=\mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x}=3 \qquad \nabla\times\mathbf{x}=0$$

$$\nabla\cdot\mathbf{n}=\frac{2}{r} \qquad \nabla\times\mathbf{n}=0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}=\frac{1}{r}[\mathbf{a}-\mathbf{n}(\mathbf{a}\cdot\mathbf{n})]\equiv\frac{\mathbf{a}_\perp}{r}$$

VECTOR CALCULUS

Cylindrical coordinates (z, r, α):

$$\nabla S = \frac{\partial S}{\partial z}\mathbf{e}^{(z)} + \frac{\partial S}{\partial r}\mathbf{e}^{(r)} + \frac{1}{r}\frac{\partial S}{\partial \alpha}\mathbf{e}^{(\alpha)}; \qquad (1)$$

$$\operatorname{div}\mathbf{v} = \frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\alpha}{\partial \alpha}; \qquad (2)$$

$$\begin{aligned} \operatorname{curl}\mathbf{v} &= \frac{1}{r}\Big[\frac{\partial}{\partial r}(r v_\alpha) - \frac{\partial v_r}{\partial \alpha}\Big]\mathbf{e}^{(z)} \\ &+ \Big[\frac{1}{r}\frac{\partial v_z}{\partial \alpha} - \frac{\partial v_\alpha}{\partial z}\Big]\mathbf{e}^{(r)} \\ &+ \Big[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\Big]\mathbf{e}^{(\alpha)}; \end{aligned} \qquad (3)$$

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \alpha^2}. \qquad (4)$$

Spherical polar coordinates (r, θ, α):

$$\nabla S = \frac{\partial S}{\partial r}\mathbf{e}^{(r)} + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}^{(\theta)} + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \alpha}\mathbf{e}^{(\alpha)}; \qquad (5)$$

$$\operatorname{div}\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\alpha}{\partial \alpha}; \qquad (6)$$

$$\begin{aligned} \operatorname{curl}\mathbf{v} &= \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial \theta}(\sin\theta\, v_\alpha) - \frac{\partial v_\theta}{\partial \alpha}\Big]\mathbf{e}^{(r)} \\ &+ \frac{1}{r}\Big[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \alpha} - \frac{\partial}{\partial r}(r v_\alpha)\Big]\mathbf{e}^{(\theta)} \\ &+ \frac{1}{r}\Big[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\Big]\mathbf{e}^{(\alpha)}; \end{aligned} \qquad (7)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\Big(\sin\theta\frac{\partial}{\partial \theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \alpha^2}. \qquad (8)$$

Expressions Involving $\sqrt{x^2 \pm a^2}$ *or* $\sqrt{a^2 - x^2}$

126. $$\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2 \log\left(x + \sqrt{x^2 \pm a^2}\right)\right]$$

127. $$\int \sqrt{a^2 - x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$$

128. $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right), \quad \text{or} \quad \sinh^{-1}\frac{x}{a}$$

129. $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right), \quad \text{or} \quad \cosh^{-1}\frac{x}{a}$$

130. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}, \quad \text{or} \quad -\cos^{-1}\frac{x}{a}$$

131. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}$$

132. $$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$$

133. $$\int \frac{\sqrt{a^2 \pm x^2}}{x}\,dx = \sqrt{a^2 \pm x^2} - a\log\frac{a + \sqrt{a^2 \pm x^2}}{x}$$

134. $$\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - a\cos^{-1}\frac{a}{x}$$

135. $$\int \frac{x\,dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}$$

136. $$\int \frac{x\,dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$$

*See Formulas 703–704.

137. $$\int x\sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2 \pm a^2)^3}$$

138. $$\int x\sqrt{a^2 - x^2}\,dx = -\tfrac{1}{3}\sqrt{(a^2 - x^2)^3}$$

139. $$\int \sqrt{(x^2 \pm a^2)^3}\,dx$$
$$= \tfrac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right)\right]$$

140. $$\int \sqrt{(a^2 - x^2)^3}\,dx$$
$$= \tfrac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{a}\right]$$

141. $$\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

142. $$\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}$$

143. $$\int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}$$

144. $$\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}$$

145. $$\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \tfrac{1}{5}\sqrt{(x^2 \pm a^2)^5}$$

146. $$\int x\sqrt{(a^2 - x^2)^3}\,dx = -\tfrac{1}{5}\sqrt{(a^2 - x^2)^5}$$

*See Formulas 703–704.

Spherical Harmonic functions

$$Y_{\ell m}(\theta,\varphi)$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta)$$

$\ell\ m$	$Y_{\ell m}(\theta\varphi)$	
0 0	$\frac{-1}{2\sqrt{\pi}}$	$\rightarrow \frac{1}{2\sqrt{\pi}}$
1 0	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\rightarrow \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$
1 ±1	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\varphi}$	$\rightarrow \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{\pm i\varphi}$
2 0	$\frac{1}{4}\sqrt{\frac{5}{\pi}}(2\cos^2\theta-\sin^2\theta)$	$\rightarrow \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(2\cos^2\theta-\sin^2\theta\right)$
2 ±1	$\mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\cos\theta\,\sin\theta e^{\pm i\varphi}$	
2 ±2	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\varphi}$	
3 0	$\frac{1}{4}\sqrt{\frac{7}{\pi}}(2\cos^3\theta-3\cos\theta\sin^2\theta)$	
3 ±1	$\mp\frac{1}{8}\sqrt{\frac{21}{\pi}}(4\cos^2\theta\sin\theta-\sin^3\theta)e^{\pm i\varphi}$	
3 ±2	$\frac{1}{4}\sqrt{\frac{105}{2\pi}}\cos\theta\sin^2\theta e^{\pm 2i\varphi}$	
3 ±3	$\mp\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta e^{\pm 3i\varphi}$	

Orthogonality:

$$\int d\Omega_{\vec n}\, Y^*_{\ell' m'}(\vec n)\, Y_{\ell m}(\vec n) = \delta_{\ell'\ell}\,\delta_{m'm} \quad ; \quad Y^*_{\ell m}(\vec n) = (-)^m\, Y_{\ell,-m}(\vec n)$$

$$\int_{-1}^{1} d\eta\, P_{\ell'}(\eta)\, P_\ell(\eta) = \frac{2}{2\ell+1}\,\delta_{\ell'\ell}$$

Addition theorem:

$$P_\ell(\cos\chi_{\vec n\vec n'}) = \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, Y^*_{\ell m}(\vec n)\, Y_{\ell m}(\vec n')$$

Multipole expansion (generating function): $\vec r = r\vec n$, $\vec r\,' = r'\vec n'$,

$$\frac{1}{|\vec r-\vec r\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}}\, P_\ell(\cos\chi_{\vec n\vec n'})$$

$$P_\ell(1) = 1, \quad P_\ell(-1) = (-1)^\ell, \quad P_\ell(0) = \frac{1+(-1)^\ell}{2}(-1)^{\ell/2}\frac{\ell!}{2^\ell\,[(\ell/2)!]^2}$$

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Final Subject Exam (Total: 50 points)
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Problem 1. A relativistic electron is moving inside a constant electromagnetic field in which the electric field $\vec{E}$ is along the positive y-axis, and the magnetic field $\vec{H}$ is along the positive z-axis. Suppose the magnitude $|\vec{E}|$ of the electric field is half of $|\vec{H}|$, and at time $t = 0$, the elctron is at the origin ($\vec{x} = 0$) with a constant velocity $v = \frac{2c}{3}$ along the positive x-axis, where c is the speed of light. We shall denote this reference frame as the k-frame.
(2 points) (a) Find the k'-frame in which either the electric or magnetic field vanishes.
(3 points) (b) Find the velocity of the electron in the k'-frame.
(2 points) (c) Find the trajectory of the electron, i.e. the position $\vec{x}'$ of the electron as a function of t', in the k'-frame.
(3 points) (d) Find the trajectory of the electron, i.e. the position $\vec{x}$ of the electron as a function of t, in the k-frame by performing Lorentz transformation on the result found in (c).

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Problem 2. The charge density inside a sphere of radius R is

$$\rho(\vec{X}) = \rho_0(2 - 3\sin^2\theta)\,, \tag{1}$$

where ρ_0 is a constant, and θ is the polar angle with respect to the z-axis that passes through the center of the sphere. (The origin is located at the center of the sphere.)
(3 points) (a) Find the total charge, the electric dipole moment and the electric quadruple moment of the charged sphere.
(7 points) (b) Find the electric potential at any point inside the charged sphere, assuming that the electric potential at the origin is zero.

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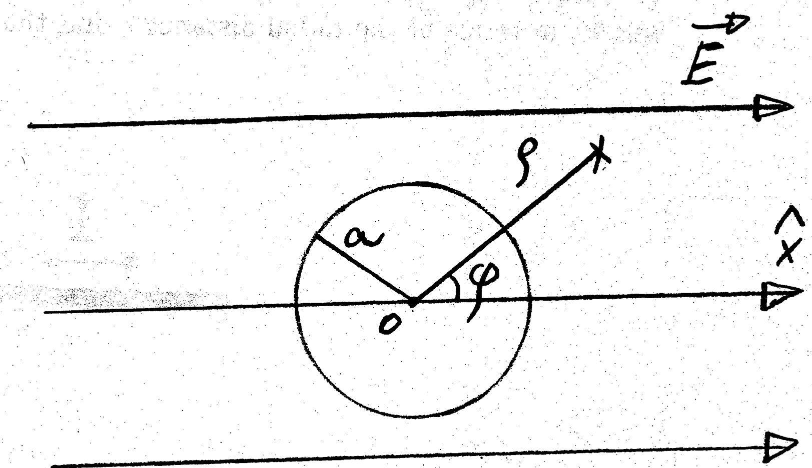
Problem 3. A long conducting cylinder (along the z-axis) of radius a is placed in an initially uniform constant electric field $\vec{E} = E\hat{x}$.
(1 points) (a) Find the initial scalar potential U_i, corresponding to the initially uniform constant electric field $\vec{E} = E\hat{x}$, at any point in space, in terms of the radial distance ρ and azimuthal angle ϕ, as shown in figure.
(2 points) (b) Find the boundary conditions for the final scalar potential U_f, which is the potential after placing the conducting cylinder.
(5 points) (c) Find the final scalar potential U_f at any point in space, in terms of ρ and ϕ.
(2 points) (d) Determine the surface charge density induced on the surface of the conducting cylinder.



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Problem 4. In a field emission microscope, charged particles (electrons) leave a hemispherical tip (radius a) of a needle-shaped cathode and move toward a concave, concentric, hemispherical anode (radius b), as shown in figure. The current in the cathode and anode is I, and the charged particles constitute an equal convection current while they are on their way from the cathode to the anode.

To simplify the problem, we assume that a is very small as compared to b, the flux density of the particles is constant at all points of the tip (cathode), and that the particles move radially toward the anode.

(3 points) (a) Determine the convection current density (per unit area).

(7 points) (b) Find the magnetic field at any point between the cathode and the anode, in terms of the radial distance r and the polar angle θ.