

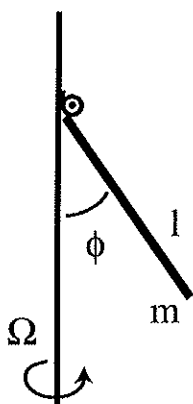
## Classical Mechanics Graduate Exam

December 10, 2003

NOTE: Do NOT write your name on any paper you hand in. Only write the selected number for the subsequent identification, so that the problems may be graded anonymously.

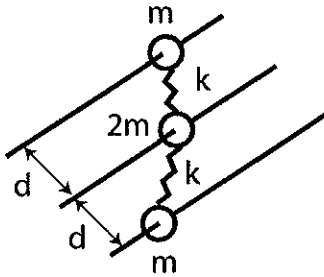
Show all work.

1. A centrifugal speedometer consists of a bar of length  $\ell$  and mass  $m$  attached, through a hinge, to a vertical axle rotated at a fixed angular velocity  $\Omega$ . As  $\Omega$  increases, on the average, the deflection angle  $\phi$  of the bar relative to the axle increases.



- (a) Find the tensor of inertia of the bar relative to the hinged end, taking one of the coordinate axes for the tensor along the bar. (b) Find the components of the net angular velocity for the bar, in terms of  $\Omega$  and  $\phi$ , in the coordinate system with one axis along the bar and another perpendicular both to the bar and the axle. (c) Obtain the kinetic energy  $T$  and Lagrangian  $L$  for the bar, using  $\phi$  as the coordinate. (d) Obtain the conserved energy function for the bar. (e) Find the equilibrium angle  $\phi$  for a given  $\Omega$ .
2. Consider two uniform solid spheres of mass  $m_1$  and  $m_2$ , and radii  $R_1$  and  $R_2$ , uniformly charged with positive charges  $Q_1$  and  $Q_2$ , respectively. Whereas sphere 1 is initially at rest, sphere 2 is shot from far away towards 1 with an initial translational energy  $T$ . (a) What is the minimum energy  $T_{min}$  required for the spheres 1 and 2 to have a chance colliding? (b) What is the total cross section for the spheres colliding at  $T > T_{min}$ ?

3. Three beads of mass  $m$ ,  $2m$  and  $m$ , respectively, are threaded onto three parallel rods, a distance  $d$  apart from each other as shown. The beads are connected with springs characterized by a spring constant  $k$ . (Assume that the length of unstretched springs is zero.) The beads can move along the rods without friction.



Find the normal modes of oscillation of the bead system (frequencies and amplitude vectors - no particular normalization required). Discuss those modes.

4. A particle of mass  $m$  moves in a potential  $V$  that, in cylindrical coordinates  $(r, \phi, z)$ , takes the form  $V = B \cos(2\phi)/r^2$ . (a) Write down a Lagrangian for the particle using cylindrical coordinates. (b) Obtain the corresponding Hamiltonian in terms of canonical variables. (c) Write down the Hamilton-Jacobi equation for the particle. (d) Solve the Hamilton-Jacobi equation, obtaining Hamilton's principal function in terms of quadratures. (e) Describe how to obtain, from Hamilton's principal function, a set of coupled equations for the dependence of  $r$ ,  $\phi$  and  $z$  on time.
5. (a) The coordinate system  $\mathcal{O}'$  rotates relative to  $\mathcal{O}$  at an angular velocity  $\vec{\omega}$ . How does the rate of change of a vector  $\vec{u}$ , as analyzed in  $\mathcal{O}'$ ,  $\left(\frac{d\vec{u}}{dt}\right)_{\mathcal{O}'}$ , relate to the rate of change in  $\mathcal{O}$ ,  $\left(\frac{d\vec{u}}{dt}\right)_{\mathcal{O}}$ ? (b) In the rigid-body frame, with axes defined by the tensor of inertia, derive the torque-free Euler's equations for the angular-velocity components, in terms of the principal values of the tensor  $I_k$  ( $k = 1, 2, 3$ ). (c) Demonstrate directly, using Euler's equations, that those equations conserve the total energy  $T$  and the magnitude of the angular momentum squared  $L^2$ , as specified in the body frame. (d) Use conservation of energy and of the magnitude of angular momentum to eliminate the body-frame angular-velocity components  $\omega_2$  and  $\omega_3$ , in terms of  $T$ ,  $L^2$ ,  $I_k$  and  $\omega_1$ , in order to obtain a closed first-order differential equation for the component  $\omega_1$ .

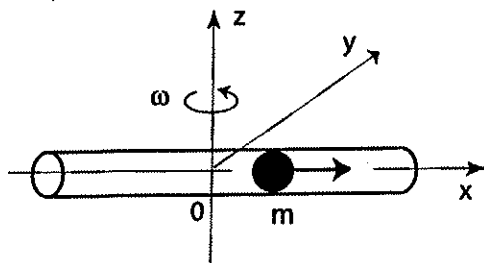
## Classical Mechanics Graduate Exam

August 25, 2003

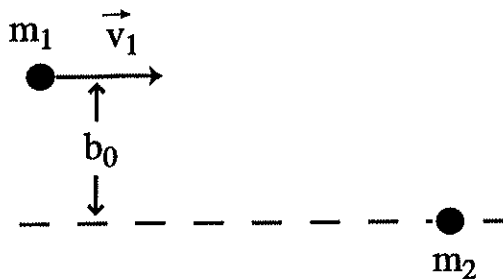
NOTE: Do NOT write your name on any paper you hand in. Only write the selected number for the subsequent identification, so that the problems may be graded anonymously.

Show all work.

1. A particle of mass  $m$  is free to move inside a horizontal tube that rotates about a vertical axis at a constant angular frequency  $\omega$ . The coefficient of friction between the particle and the tube is  $\mu$ . Effects of gravity may be ignored.

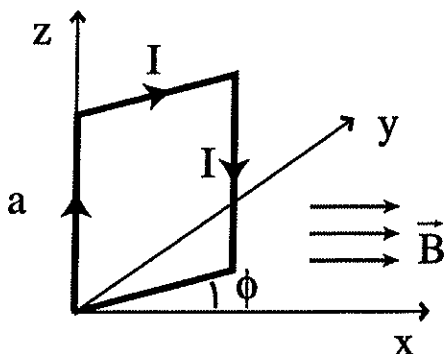


- (a) In the noninertial frame associated with the tube, find all the forces on the particle, no matter whether real or effective. (b) Obtain and solve an equation of motion for the particle in the coordinate  $x$  specifying the particle position along the tube. (c) Determine and discuss the solution of the equation for the starting condition of particle at rest at position  $x = x_0$ , at time  $t = 0$ .
2. Two particles of masses  $m_1$  and  $m_2$  ( $m_1 \neq m_2$ ) collide. The initial velocity of particle 1 is  $\vec{v}_1$ , while the particle 2 is initially at rest. The initial impact parameter is  $b_0$ , as shown.



The particles interact through a repulsive potential  $V = V_0/|\vec{r}_1 - \vec{r}_2|^4$ . (a) Find the magnitude of net angular momentum and the net energy in the *center of mass*, in terms of provided quantities. (b) Obtain an equation for the rate of change of the separation  $r = |\vec{r}_1 - \vec{r}_2|$  in time. Find the distance of closest approach between the particles. (c) Consider a situation where  $b_0$  is unknown but the magnitude  $v_1^f$  of the final velocity of particle 1 has been measured in the laboratory frame. Find the angle  $\beta$  between the final laboratory velocities of particles 1 and 2, given  $v_1^f$ . (Use conservation laws whenever possible.)

3. An ideally conductive square loop can rotate around its side placed on the  $z$ -axis, as shown, within a constant uniform magnetic field  $\vec{B}$  along the  $x$ -axis.



The loop's side length is  $a$ , moment of inertia is  $J$  and self-inductance is  $\mathcal{L}$ . As generalized coordinates describing the loop, one can use the angle  $\phi$  of the loop relative to the  $x$ -axis and the net charge  $q$  that passed around the loop in the clockwise direction. (The current is  $I = \dot{q}$ .) In terms of those coordinates, the Lagrangian for the loop can be written as

$$L(\phi, \dot{\phi}, q, \dot{q}) = \frac{1}{2} J \dot{\phi}^2 + \frac{1}{2} \mathcal{L} \dot{q}^2 - \dot{q} a^2 B \sin \phi.$$

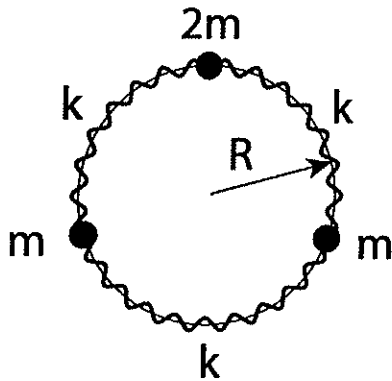
Here, one can recognize the rotational and inductive energies of the loop and an interaction term of the loop's magnetic moment with the field. (a) From the Lagrangian, find the conserved quantities for the motion of the loop. Can you interpret those quantities? (b) Obtain a Hamiltonian for the loop in terms of the specified coordinates and generalized momenta. (c) Exploit the conservation laws from (a) to obtain an effective potential  $U_{eff}(\phi)$  for the motion of the loop in  $\phi$ . Sketch the potential and discuss qualitatively the possible motions in  $\phi$  depending on initial conditions.

4. Consider small oscillations of an anharmonic oscillator, with the Hamilton's function given by:

$$H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \alpha q^3,$$

where  $\alpha q \ll \omega^2$ . (a) Apply a canonical transformation, following the generating function  $\Phi = qP + aq^2P + bP^3$ , with the small constants  $a$  and  $b$  adjusted to eliminate any anharmonic terms from the the Hamilton's function up to the first order in  $\alpha Q/\omega^2$ . (b) From the solution of the Hamilton's equations in  $Q$  and  $P$ , find a general form of  $q(t)$  for this oscillator.

5. Three beads are mounted on a ring and connected by three identical springs, as shown.



Two beads are of mass  $m$  and one of mass  $2m$ . The ring radius is  $R$ . The spring constant is  $k$  and the spring mass may be neglected. The masses and springs are free to move around the ring. (a) Find a Lagrangian for the system of beads and springs, in terms of suitably chosen coordinates. (b) Find frequencies of normal vibrations for the system. (c) Find normal coordinates for the system. (Can you find any shortcuts, bypassing standard procedures, relying on physical or symmetry considerations?)



# CLASSICAL MECHANICS SUBJECT EXAM

August 27, 2001

5:10 PM - 8:10 PM

**DO NOT WRITE YOUR NAME ON ANY PAPER YOU HAND IN. ONLY WRITE YOUR CHOSEN NUMBER FOR SUBSEQUENT IDENTIFICATION, SO THE PROBLEMS MAY BE GRADED ANONYMOUSLY BY THREE FACULTY. SHOW ALL YOUR WORK.**

## PROBLEM 1 10 POINTS

*A body is dropped from rest at a height  $h$  above the surface of the earth and at a latitude  $60^\circ$  N. For  $h = 1$  kilometer, calculate the lateral displacement of the point of impact due to the Coriolis force alone [Since the time of the drop to the earth's surface is short compared with the period of rotation of the earth, ignore terms of order  $\omega^2$ .]*

**PROBLEM 2 10 POINTS**

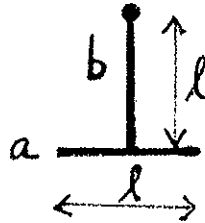
*A particle of mass  $m$  moves freely with constant initial velocity  $\vec{v}_1$  and speed  $v_1$  in a medium #1 towards a plane interface boundary with another medium #2. The potential energy  $U$  takes constant but different respective values  $U_1$  and  $U_2$  on the two sides of the plane interface boundary. The initial velocity  $\vec{v}_1$  makes an angle of incidence  $\theta_1$  with the normal to the plane interface boundary.*

*Find the changed velocity  $\vec{v}_2$  and speed  $v_2$  possessed by the particle when it passes through the plane interface boundary, specifying explicitly the changed direction of motion by calculating the angle  $\theta_2$  made by the particle velocity with the normal to the plane interface boundary in medium #2. For what conditions will the particle be unable to penetrate into the second medium ?*



**PROBLEM 3 10 POINTS**

A pendulum is constructed from two identical uniform thin rods *a* and *b*, each of length *l* and mass *m*, connected at right angles to form a "T" by joining the center of rod *a* to one end of rod *b*, as illustrated below. The "T" is then suspended from the free end of rod *b*. The pendulum swings in the plane of the "T".



- a) 2 points. Calculate the moment of inertia  $I$  of the "T" about the axis of rotation.
- b) 2 points. Obtain expressions for the kinetic and potential energies in a uniform gravitational field  $g$  in terms of the angle  $\theta$  of inclination to the vertical of the pendulum.
- c) 4 points. Derive the equation of motion of the pendulum.
- d) 2 points. Derive the period of small oscillations.

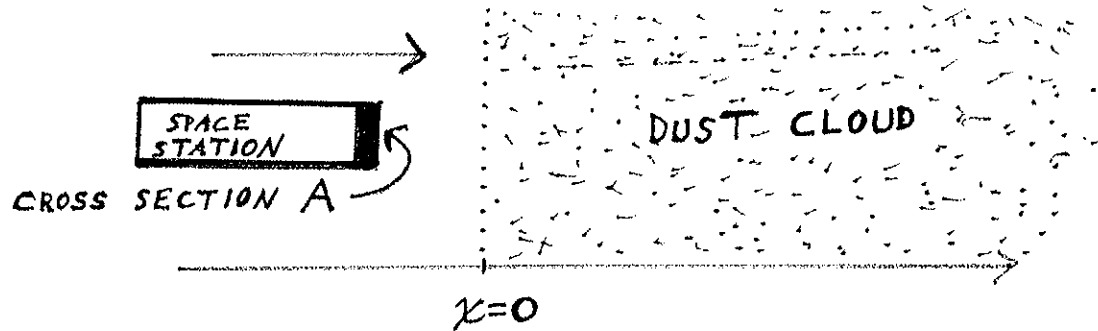
**PROBLEM 4 10 POINTS**

*A mass  $m$  slides on a horizontal frictionless track while it remains connected to the end of a spring whose other end is fixed on the track. Initially, the amplitude of the oscillations is  $A_1$  and the spring constant is  $k_1$ . But a mist or cloud of nitric acid droplets comes over the spring and slowly dissolves it, and decreases adiabatically the spring constant at a uniform rate until the value  $k_2$  is reached. Calculate the new amplitude of oscillation.*

**PROBLEM 5 10 POINTS**

Suppose a space station of mass  $m_0$  and cross-sectional area  $A$ , as illustrated below, is moving with velocity  $v_0$  when it suddenly enters a stationary dust cloud in space of uniform mass density  $\rho$ . Assume that the dust particles stick to the front surface of the space station, and that  $A$  is constant over time, and only becomes thicker.

Obtain an expression for the subsequent motion  $x(t)$  of the space station, after it enters the dust cloud.





**Instructions:** The test emphasizes basic concepts, rather than detailed calculations. Spell out the reasoning and the procedures clearly. Use illustrations whenever appropriate, both to help your own thinking, and to demonstrate to the graders that you understand the problem qualitatively. Do not get bogged down on some particular problem. Use the available time wisely. The wide range of the problems is meant to offer everyone a chance to show his/her knowledge and ability. Do as many problems as you can. Although it is certainly possible to complete all the problems in the allotted time, such a performance would be quite exceptional.

1. Two identical simple harmonic oscillators are coupled in such a way that the Lagrangian for the whole system is  $L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}(x_1^2 + x_2^2) + \alpha x_1 x_2$ , where  $|\alpha| < 1$  and we have chosen the spring constant to be unity for convenience.
  - (a) Find the normal coordinates and the eigen-frequencies of this coupled system.
  - (b) Characterize (i.e. describe in qualitative terms) the motion of the two oscillating particles in each of these normal modes.
  - (c) If the system starts at  $t = 0$  with the initial conditions  $\{x_1 = A; x_2 = 0; \text{ and } \dot{x}_1 = \dot{x}_2 = 0\}$ , will the motion of the system be in normal mode 1, or 2, or a combination of both? If the latter, what is the ratio of the amplitudes of the two normal modes?
  
2. Two skaters G(eorge) and H(arry), both of mass  $M = 70$  kg, are approaching one another, each with a speed of  $v_0 = 2$  m/s. G carries a bowling ball with a mass of  $m = 10$  kg.
  - (i) G tosses the ball toward H at  $u = 4$  m/s (relative to the thrower) when they are  $D_0 = 30$  m apart.
    - (a) While the Ball (B) is in the air, what are the velocities (magnitudes and signs) of G, B and H ( $v_1^G, v_1^B$  and  $v_1^H$ )?
    - (b) Show graphically the time development of the positions of G, B and H (i.e. plot the positions  $x_{G,B,H}$  vs.  $t$ ), let  $x_{G,H}(t = 0) = \mp 15$  m respectively.
  - (ii) H catches the ball and immediately tosses it back toward G with the same relative speed  $u$  (with respect to the thrower).
    - (a) What are the velocities of G, B, and H when the ball is in the air this time, ( $v_2^G, v_2^B, v_2^H$ )?
    - (b) Plot this motion on the above graph (i.b). Assuming the two skaters continue to throw the ball back and forth, how will their motion develop according to this graph? (Will they collide?)

(This example is often used as a model of a repulsive *exchange force* between two objects (G and H) mediated by a third particle (B)).
  
3. Discuss the 2-dimensional motion of a particle moving in an attractive central-force described by the force law  $F(r) = -k/r^\alpha$ , where  $k$  is positive and  $3 > \alpha > 2$ .
  - (a) Write down the equations of motion in polar coordinates;
  - (b) Show how conservation laws can be used to derive the formal equation for the orbit of motion;
  - (c) Describe the nature of the orbits for various possible initial energies and angular momenta. (Graphical methods can be very useful.)

4. A steel cylinder of mass  $M$ , radius  $R$ , and moment of inertia  $I = \frac{1}{2}MR^2$  rests across the width of a flatbed truck. The flat bed floor has length  $L$ .
- (i) As the truck accelerates at a constant acceleration of  $a = \frac{1}{2}g$  in the positive  $x$  direction, the pipe starts to roll without slipping (due to *static friction* force).
- Write down the condition for rolling without slipping. (Use the angle of rotation  $\theta$ , and linear coordinate of the center of the pipe  $x$  with respect to the ground as variables.)
  - Plot qualitatively the velocities  $v = \dot{x}$  and  $\omega = \dot{\theta}$  as a function of time. (It is useful to plot  $v$  and  $\omega R$  alongside each other on the same plot.)
  - What are the linear and angular velocities (magnitude and direction) of the pipe just before it falls off from the truck?
- (ii) After the pipe hits the ground, its motion will, at first, consist of rolling *and* skidding.
- If the coefficient of kinetic friction between the moving cylinder and the level ground is  $\mu_k = \frac{1}{4}$ , what are the linear and angular accelerations ( $\dot{v}, \dot{\omega}$ ) while the cylinder is skidding? Use these results to extend the velocity plot of part (i.b) above.
  - When the cylinder ceases to skid, what are its linear and angular velocities? (The above graph can be quite helpful here.)
5. Consider a charged particle of mass  $m$  moving in a constant magnetic field of magnitude  $B$  in the  $z$  direction. The motion can be described by a Lagrangian such as  $L = \frac{1}{2}m\dot{v}^2 + eBxv_y$ .
- Write down the corresponding Hamiltonian for this system in terms of an appropriate set of canonical variables.
  - Write down the Hamilton-Jacobi equation for this system.
  - Outline the procedure that can be used to solve this equation.

# CLASSICAL MECHANICS SUBJECT EXAM

MONDAY, AUGUST 28, 2000  
5:10 – 8:10 PM, ROOM 317 PA

**DO NOT WRITE YOUR NAME ON ANY PAPER YOU HAND IN. ONLY WRITE YOUR CHOSEN NUMBER FOR SUBSEQUENT IDENTIFICATION, SO THE PROBLEMS MAY BE GRADED ANONYMOUSLY BY THREE FACULTY. SHOW ALL YOUR WORK.**

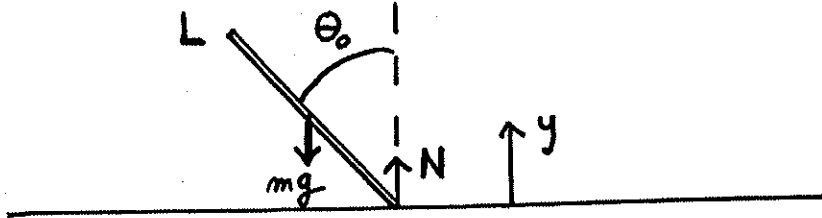
Problem 1. 10 points.

The captain of a small boat in the region of the equator notices that there is no wind in the sails. The boat is at rest in very shallow water. Because the captain understands physics, he makes an attempt to move the boat in the water by slowly raising the anchor of mass  $m$  to the top of a mast a distance  $s$  above the surface of the water. The rest of the boat, including the crew, has a mass  $M \gg m$ .

- a) 3 points Why will the boat begin to move?
- b) 2 points In which direction will it move? [Recall that the angular velocity of the earth is a vector that points north. The rotating earth is not an inertial frame.]
- c) 5 points Derive an expression for the velocity of the boat relative to the water, as a function of  $s$ , while the anchor is raised with velocity  $v$ .

**Problem 2. 10 points**

**A thin uniform stick of mass  $m$ , with its bottom end resting on a frictionless table, is released from rest at an angle  $\theta_0$  to the vertical:**



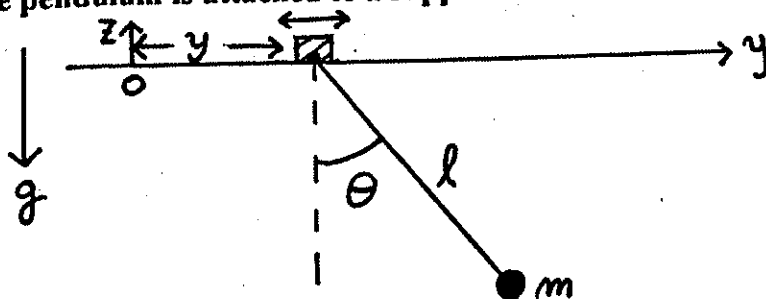
**10 points**

**Calculate the force exerted by the table upon the stick immediately after the release.**



**Problem 3. 10 points**

**A simple pendulum is attached to a support which is driven horizontally with time:**



**The simple pendulum has length  $l$  and it has mass  $m$  attached at its unsupported end.**

- a) 2 points      **Obtain the Lagrangian for the system in terms of the generalized coordinates  $\theta$  and  $y$ , where  $\theta$  is the angular displacement from equilibrium and  $y(t)$  is the horizontal position of the pendulum support.**
- b) 3 points      **Derive the Euler-Lagrange equation of motion for  $\theta$ .**
- c) 5 points      **For angular displacements which remain small and a sinusoidal motion of the moving support  $y=y_0 \cos(\omega t)$ , find the steady-state general solution to the equation of motion which reflects any initial conditions that may have been present.**

**Problem 5. 10 points**

A particle of mass  $m$  slides under the influence of gravity on the inside of a smooth paraboloid of revolution whose axis is vertical:  $z = A r^2$ ,  $A > 0$ . Using the distance from the axis  $r$  and the azimuthal angle  $\phi$  as generalized coordinates, find:

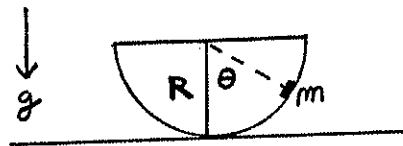
- a) 2 points      The Lagrangian of the system.
- b) 3 points      The generalized momenta and the corresponding Hamiltonian.
- c) 3 points      The equation of motion for the coordinate  $r$  as a function of time.
- d) 2 points      If  $d\phi/dt = 0$ , show that the particle can execute small oscillations about the lowest point of the paraboloid. Find the angular frequency of these oscillations.

# CLASSICAL MECHANICS SUBJECT EXAM

WEDNESDAY, DECEMBER 13, 2000  
7:45 – 10:45 AM, ROOM 317 PA

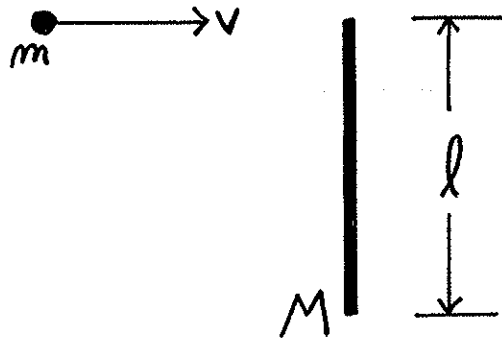
**DO NOT WRITE YOUR NAME ON ANY PAPER YOU HAND IN. ONLY WRITE YOUR CHOSEN NUMBER FOR SUBSEQUENT IDENTIFICATION, SO THE PROBLEMS MAY BE GRADED ANONYMOUSLY BY THREE FACULTY. SHOW ALL YOUR WORK.**

A bowl of hemispherical shape with radius  $R$  is resting in a fixed position in a gravitational field  $g$ , with its open circular rim in a horizontal plane, as shown. A small mass  $m$  on the rim of the bowl begins to slide down the bowl without friction from an initial situation of rest on the rim. Calculate how much time it takes for the mass to return to its initial position after beginning to slide. You may leave your answer in terms of a definite integral.



**Problem 2. 10 points**

A particle of known mass  $m$  and speed  $v$  is moving perpendicular to a uniform thin rod of unknown mass  $M$  and length  $l$ , which is at rest. If the particle collides elastically with the end of the rod, and after the collision the particle is stationary, calculate  $M$ . [Hint: Recall elementary conservation theorems of Newtonian mechanics for an isolated system.]



**Problem 3. 10 points**

**a) - 4 points.**

A particle of mass  $m$  moves under a conservative force with potential energy  $V(x) = cx / (x^2 + a^2)$ , where  $c$  and  $a$  are positive constants. Find the position of stable Equilibrium and the period of small oscillations about it.

**b) - 6 points.**

If the particle starts from the stable equilibrium point with velocity  $v$ , find the range of values of  $v$  for which it (1) oscillates, (2) escapes to  $-\infty$ , (3) escapes to  $+\infty$ .

**Problem 4. 10 points**

The Lagrangian for a particle of mass  $m$  and charge  $e$  moving in a static electromagnetic field is

$$L = (m/2) \dot{\vec{x}}^2 + e \dot{\vec{x}} \cdot \vec{A}(\vec{x}) - e \varphi(\vec{x}),$$

Where  $\vec{A}(\vec{x})$  is the vector potential, related to the magnetic field as  $\vec{B} = \nabla \times \vec{A}(\vec{x})$ , and  $\varphi(\vec{x})$  is the electric potential, related to the electric field as  $\vec{E} = -\nabla \varphi(\vec{x})$ .

a) - 2 points

Derive expressions for all three components of the canonical momentum  $\vec{p}$  conjugate to the coordinate components of the position vector  $\vec{x}$ .

b) - 3 points

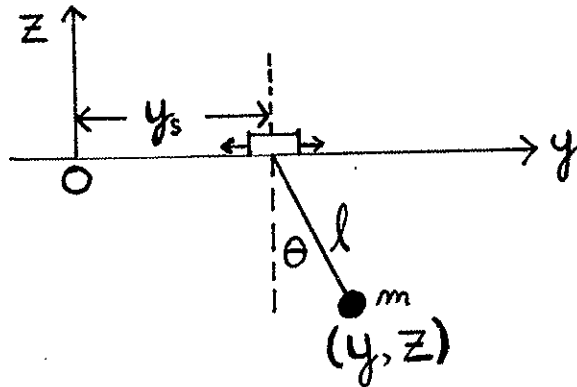
Derive the Hamiltonian  $H$  and express it in terms of the canonical variables defined by the components of the vectors  $\vec{x}$  and  $\vec{p}$ .

c) - 5 points

Consider the case where  $\varphi(\vec{x}) = 0$  and  $\vec{A}(\vec{x}) = (A_x, A_y, A_z) = (0, xB, 0)$ , where  $B$  is the magnitude of a constant magnetic field  $\vec{B}$  in the  $z$  direction. Use Hamilton's equations to derive expressions for the  $x$  and  $y$  components of the particle's acceleration.

**Problem 5. 10 points.**

A simple pendulum of length  $l$  with mass  $m$  at the end is attached to a moving support which is driven horizontally with time as shown below.



a) - 3 points.

Set up the Lagrangian for the system in terms of the generalized coordinates  $\theta$  and mass coordinate  $y$ , where  $\theta$  is the angular displacement from the vertical and  $y_s(t)$  is the prescribed horizontal position of the pendulum support.

b) - 4 points.

Derive the equation of motion for  $\theta(t)$ .

c) - 3 points.

For small angular displacements and a sinusoidal motion of the support  $y_s = Y_0 \cos(\omega t)$ , solve for the general steady-state solution to the equation of motion for  $\theta(t)$ .

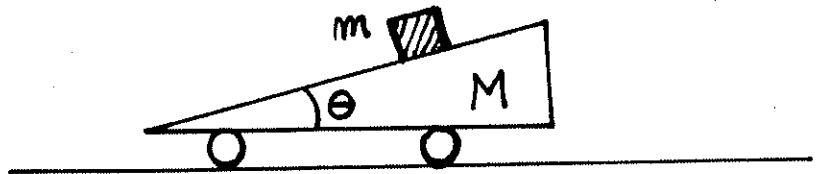
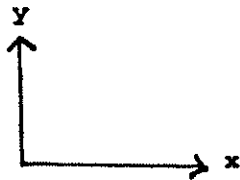




**DO NOT WRITE YOUR NAME ON ANY PAPER YOU HAND IN. ONLY WRITE YOUR CHOSEN NUMBER FOR SUBSEQUENT IDENTIFICATION, SO THE PROBLEMS MAY BE GRADED ANONYMOUSLY BY THREE FACULTY. SHOW ALL YOUR WORK.**

PROBLEM 1 10 points

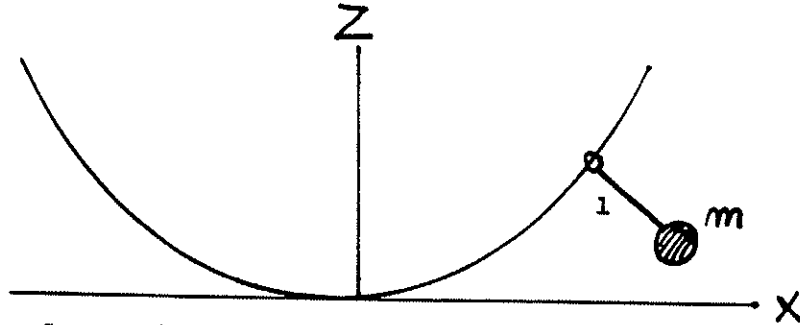
An iron cube of mass  $m$  slides without friction on a plane surface inclined at an angle  $\theta$  to the horizontal in a vertical uniform gravitational field  $g$ . The plane itself is on massless rollers and is free to move horizontally, also without dissipative friction of any kind. The plane has mass  $M$ .



- a) 5 points. Calculate the vector components of the acceleration  $\vec{A}$  of the plane.
- b) 5 points. Calculate the vector components of the acceleration  $\vec{a}$  of the mass  $m$ .

**PROBLEM 2** 10 points

A simple pendulum consists of a massless rigid rod of length  $l$  with a mass  $m$  attached to the end of the rod. The point of suspension of the rod is constrained to move on a parabola  $Z = AX^2$  in the vertical plane in the gravitational field  $g$ .



- a) 4 points. Derive a Lagrangian governing the motion of the pendulum AND its point of suspension in the  $(X,Z)$  plane.
- b) 4 points. Find conjugate momenta for whatever generalized coordinates you have chosen.
- c) 2 points Obtain Euler-Lagrange equations describing the system's motion.

**PROBLEM 3** 10 points

A particle of mass  $m$  moving in one dimension is attracted to a fixed point with a conservative force having a magnitude  $|F| = k s^3$ , where  $s$  is the distance from the fixed point.

- a) 2 points. Choose a coordinate system and write down a formula that expresses the total energy of the particle.
- b) 6 points. From part a) obtain an expression for the frequency of the particle motion when it is released from rest a distance  $S_0$  from the attracting fixed point. Exhibit the explicit  $S_0$ -dependence of that frequency.
- c) 2 points. If the release distance is now reduced from  $S_0$  to  $S_0/2$ , what is the ratio of the new frequency to the old frequency?

PROBLEM 4 10 points

A particle of mass  $m$  is attracted to the origin with a central force given by  $\vec{F} = -k \vec{r}$ , where  $k$  is a constant, but it is confined to move in the X-Y plane, where  $Z=0$ .

- a) 1 point. Calculate the potential in the plane of motion in terms of a choice of coordinates that explicitly exhibits its symmetry.
- b) 3 points. Derive a Lagrangian for the particle using the potential obtained in a), and calculate the conjugate momenta for the particle.
- c) 4 points. Calculate the Hamiltonian for the particle in terms of the canonical coordinates and conjugate momenta.
- d) 2 points. Derive Hamilton's equations of motion for the canonical coordinates and momenta, and solve them for the coordinates' dependence on time, in terms of exact analytical expressions or indefinite integrals. Remember that a particle moving in a plane has two degrees of freedom, so that your solution for the time dependent coordinates should contain 4 arbitrary constants.

**PROBLEM 5**    10 points

Consider a one-dimensional simple harmonic oscillator of mass  $m$  and force constant  $k$  driven by a sinusoidal force  $F = \text{Real part of } F_0 e^{i\omega t}$ , where  $F_0$  is some complex amplitude constant,  $t$  is the time, and  $\omega$  is the angular frequency of the driving force. The motion of the oscillating particle is damped by a frictional force  $f = -2m\gamma v$ , where  $v$  is the directed velocity of the particle and  $\gamma$  is a positive constant.

- a) 3 points. Calculate the resonant frequency, where the amplitude of the response to  $F$  is greatest, taking into account the effect of the frictional force.
- b) 2 points. Calculate the phase angle between the oscillator displacement and the driving force in the steady state.
- c) 3 points. Calculate the average power absorbed by the moving particle from the applied force  $F$ , as a function of its angular frequency.
- d) 2 points. Calculate the maximum power absorbed, at the resonant frequency, and the width, in frequency, at half-maximum of the curve giving the average power absorbed [the function calculated in part c)].



**CLASSICAL MECHANICS SUBJECT EXAM**

**August 30, 1999**

**Problem 1.** A uniform rectangular platform of length  $L$  hangs by two ropes making angles  $\theta_1$  and  $\theta_2$  to the vertical. A point mass load of twice the mass of the platform is placed on it to keep it horizontal.

**(10 points)** Calculate how far from the left-hand edge of the platform it must be.

**CLASSICAL MECHANICS SUBJECT EXAM**  
**August 30, 1999**

**Problem 2.** A circular wire of radius  $R$  rotates with constant angular velocity  $\Omega$  about a vertical diameter in the presence of gravity  $g$ . A small bead of mass  $m$  is free to move without friction on the circumference, as shown.

- (4 points)** Choose independent coordinates for the bead, and write down a Lagrangian.
- (2 points)** Write down Lagrange's Equation, and derive the equation of motion for the bead.
- (4 points)** Find and specify the locations on the rotating circle where the bead will remain, if initially placed there with zero speed along the wire.



**CLASSICAL MECHANICS SUBJECT EXAM**

**August 30, 1999**

**Problem 3.** Consider a one-dimensional simple harmonic oscillator driven by a sinusoidal force  $F = F_0 \cos(\omega t)$  and damped by a frictional force  $f = -2m\gamma v$ , where  $m$  is the mass of the moving object,  $v$  is its velocity, and  $\gamma$  is a positive constant.

**(5 points)** If  $k$  is the force constant characteristic of the (undamped and undriven) simple harmonic oscillator, calculate the resonant frequency taking into account the effect of the frictional force.

**(5 points)** Calculate the phase angle between the oscillator displacement and the driving force in a steady state.

**SHOW ALL YOUR WORK.**

**CLASSICAL MECHANICS SUBJECT EXAM**  
**August 30, 1999**

**Problem 4.** Consider a particle scattered by a fixed attractive central potential  $U(r)$  centered at  $r=0$ . For an incident particle with finite angular momentum, it is still possible for the particle to reach the origin at  $r=0$  provided certain conditions are met.

**(10 points)** If  $U(r) = -\alpha/r^2$ , calculate the effective cross-section  $\sigma$  for the specific process only in which an incident particle reaches the origin when it had an initial incident speed  $v_\infty$  at great distance.

**CLASSICAL MECHANICS SUBJECT EXAM**  
**August 30, 1999**

**Problem 5.** A uniformly dense triangular block of mass  $m_1$ , length  $a$ , and height  $b$  is supported by two uniformly dense circular cylinders of mass  $m_2$  and radius  $r_2$  that roll without slipping on a horizontal surface. A uniformly dense sphere of mass  $m_3$  and radius  $r_3$  is placed on top as shown. The sphere also rolls without slipping. The whole apparatus is released from rest.

**(5 points)** Write down two independent constants of the motion.

**(5 points)** Write down a Lagrangian, and use it to find the equations of motion for the system.

**As coordinates, use the horizontal position of the block  $x_1$ , and the horizontal position of the center of the sphere  $x_3$ .**



# CLASSICAL MECHANICS SUBJECT EXAM

August 30, 1999

**Problem 1.** A uniform rectangular platform of length  $L$  hangs by two ropes making angles  $\theta_1$  and  $\theta_2$  to the vertical. A point mass load of twice the mass of the platform is placed on it to keep it horizontal.

**(10 points)** Calculate how far from the left-hand edge of the platform it must be.

**CLASSICAL MECHANICS SUBJECT EXAM**  
**August 30, 1999**

**Problem 2.** A circular wire of radius  $R$  rotates with constant angular velocity  $\Omega$  about a vertical diameter in the presence of gravity  $g$ . A small bead of mass  $m$  is free to move without friction on the circumference, as shown.

- (4 points)** Choose independent coordinates for the bead, and write down a Lagrangian.
- (2 points)** Write down Lagrange's Equation, and derive the equation of motion for the bead.
- (4 points)** Find and specify the locations on the rotating circle where the bead will remain, if initially placed there with zero speed along the wire.

**CLASSICAL MECHANICS SUBJECT EXAM**

**August 30, 1999**

**Problem 3.** Consider a one-dimensional simple harmonic oscillator driven by a sinusoidal force  $F = F_0 \cos(\omega t)$  and damped by a frictional force  $f = -2m\gamma v$ , where  $m$  is the mass of the moving object,  $v$  is its velocity, and  $\gamma$  is a positive constant.

**(5 points)** If  $k$  is the force constant characteristic of the (undamped and undriven) simple harmonic oscillator, calculate the resonant frequency taking into account the effect of the frictional force.

**(5 points)** Calculate the phase angle between the oscillator displacement and the driving force in a steady state.

**SHOW ALL YOUR WORK.**

**CLASSICAL MECHANICS SUBJECT EXAM**  
**August 30, 1999**

- Problem 4.** Consider a particle scattered by a fixed attractive central potential  $U(r)$  centered at  $r=0$ . For an incident particle with finite angular momentum, it is still possible for the particle to reach the origin at  $r=0$  provided certain conditions are met.
- (10 points)** If  $U(r) = -\alpha/r^2$ , calculate the effective cross-section  $\sigma$  for the specific process only in which an incident particle reaches the origin when it had an initial incident speed  $v_\infty$  at great distance.



# CLASSICAL MECHANICS SUBJECT EXAM

August 30, 1999

**Problem 5.** A uniformly dense triangular block of mass  $m_1$ , length  $a$ , and height  $b$  is supported by two uniformly dense circular cylinders of mass  $m_2$  and radius  $r_2$  that roll without slipping on a horizontal surface. A uniformly dense sphere of mass  $m_3$  and radius  $r_3$  is placed on top as shown. The sphere also rolls without slipping. The whole apparatus is released from rest.

**(5 points)** Write down two independent constants of the motion.

**(5 points)** Write down a Lagrangian, and use it to find the equations of motion for the system.

As coordinates, use the horizontal position of the block  $x_1$ , and the horizontal position of the center of the sphere  $x_3$ .



PHY-820 - CLASSICAL MECHANICS

Final Exam

**Problem 1 - 10 Points**

A homogeneous cube of side length  $l$  and mass  $m$  is initially at rest in unstable equilibrium standing on one of its edges touching the horizontal plane. It then begins to tip over from this position.

- a) Determine the moment of inertia of the cube around an axis parallel to an edge and through its center of mass. Remember that the elements of the tensor are given by  $I_{ij} = \int_V dV \rho(\vec{r}) \cdot (\vec{r}^2 \delta_{ij} - r_i r_j)$  [2 pts]
- b) Using symmetry arguments, determine the entire inertia tensor [2 pts]
- c) Assume the edge in contact with the plane cannot slide. Determine the moment of inertia around that edge. What is the angular velocity at the time the face of the cube contacts the plane? [3 pts]
- d) Assume the plane is frictionless so that the edge in contact with the plane can slide, and so that the rotation occurs around the cube's center axis. What is the angular velocity at the time the face of the cube contacts the plane? [3 pts]

**Problem 2 - 10 Points**

Consider the 2-dimensional motion of a particle moving in an attractive central force  $F(r) = -k/r^\alpha$ , where  $2 < \alpha < 3$ .

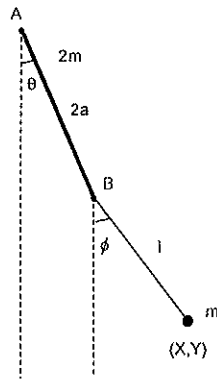
- a) Determine the Lagrangian and the equations of motion in polar coordinates. [2 pts]
- b) Determine the generalized momenta and the Hamiltonian. [2 pts]
- c) Show that one of the generalized momenta is conserved. Is the Hamiltonian conserved? [2 pts]
- d) Utilizing that one of the momenta is conserved, re-write the equations of motion of the other variable so that it becomes one dimensional. [2 pts]
- e) Without attempting to solve the problem, qualitatively classify the various cases of solutions depending on the energy of the system. A sketch may be helpful. [2 pts]

**Problem 3 - 10 Points**

Consider a double pendulum swinging in the plane as follows. A uniform rod of length  $2a$  and mass  $2m$  is attached to pivot around one of its ends. At the other end of the rod, a string of length  $l$  is attached, at the end of which there is a point mass  $m$ .

a) Determine the Lagrangian of the system in terms of the angles  $\theta$  and  $\phi$  between the rod and string and the vertical, respectively, keeping terms up to second order in the variables. [6 pts]

b) Determine the equations of motion for  $\ddot{\theta}$  and  $\ddot{\phi}$  [4 pts]



**Problem 4 - 10 Points**

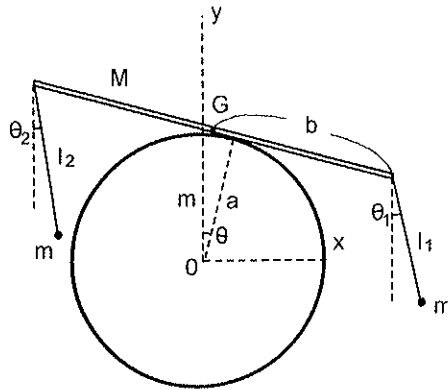
A thin uniform rod of mass  $M$  and length  $2b$  is placed on top of a stationary cylinder of radius  $a$ , so that the center of mass of the rod rests on the uppermost part of the cylinder. The cylinder is attached to the ground and does not move. On both ends of the rod, masses  $m$  are attached by strings of lengths  $l_1$  and  $l_2$ . The setup is then moved from equilibrium to perform small oscillations. The contact between the cylinder and the rod is assumed to be dominated by friction, i.e. the rod does not slide over the cylinder's surface.

a) Determine the Cartesian position of the center of mass of the rod as a function of its tilt angle  $\theta$ . Also determine the Cartesian positions of the two masses as a function of the angles  $\theta_1$  and  $\theta_2$  of their respective strings. [3 pts]

b) Determine the Cartesian velocities of the three locations in part a) as a function of  $\theta$ ,  $\theta_1$  and  $\theta_2$ . [2 pts]

c) Perform a small angle approximation of the velocities and determine the Lagrangian under this approximation. [3 pts]

d) Determine the resulting equations of motion and the frequencies of oscillations. [2 pts]



PHY-820 - CLASSICAL MECHANICS

Final Exam

**Problem 1 - 10 Points**

A homogeneous cube of side length  $l$  and mass  $m$  is initially at rest in unstable equilibrium standing on one of its edges touching the horizontal plane. It then begins to tip over from this position.

- a) Determine the moment of inertia of the cube around an axis parallel to an edge and through its center of mass. Remember that the elements of the tensor are given by  $I_{ij} = \int_V dV \rho(\vec{r}) \cdot (\vec{r}^2 \delta_{ij} - r_i r_j)$  [2 pts]
- b) Using symmetry arguments, determine the entire inertia tensor [2 pts]
- c) Assume the edge in contact with the plane cannot slide. Determine the moment of inertia around that edge. What is the angular velocity at the time the face of the cube contacts the plane? [3 pts]
- d) Assume the plane is frictionless so that the edge in contact with the plane can slide, and so that the rotation occurs around the cube's center axis. What is the angular velocity at the time the face of the cube contacts the plane? [3 pts]

**Problem 2 - 10 Points**

Consider the 2-dimensional motion of a particle moving in an attractive central force  $F(r) = -k/r^\alpha$ , where  $2 < \alpha < 3$ .

- a) Determine the Lagrangian and the equations of motion in polar coordinates. [2 pts]
- b) Determine the generalized momenta and the Hamiltonian. [2 pts]
- c) Show that one of the generalized momenta is conserved. Is the Hamiltonian conserved? [2 pts]
- d) Utilizing that one of the momenta is conserved, re-write the equations of motion of the other variable so that it becomes one dimensional. [2 pts]
- e) Without attempting to solve the problem, qualitatively classify the various cases of solutions depending on the energy of the system. A sketch may be helpful. [2 pts]

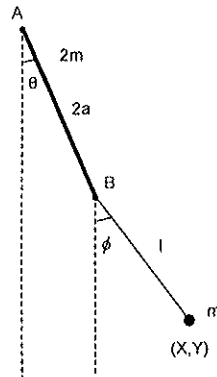


**Problem 3 - 10 Points**

Consider a double pendulum swinging in the plane as follows. A uniform rod of length  $2a$  and mass  $2m$  is attached to pivot around one of its ends. At the other end of the rod, a string of length  $l$  is attached, at the end of which there is a point mass  $m$ .

a) Determine the Lagrangian of the system in terms of the angles  $\theta$  and  $\phi$  between the rod and string and the vertical, respectively, keeping terms up to second order in the variables. [6 pts]

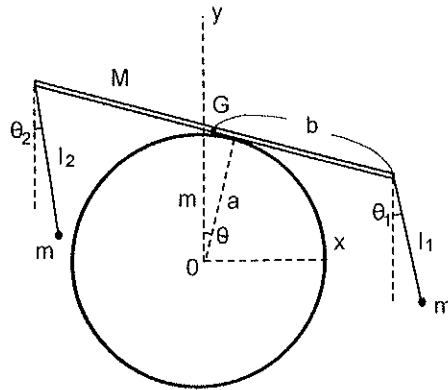
b) Determine the equations of motion for  $\ddot{\theta}$  and  $\ddot{\phi}$  [4 pts]



**Problem 4 - 10 Points**

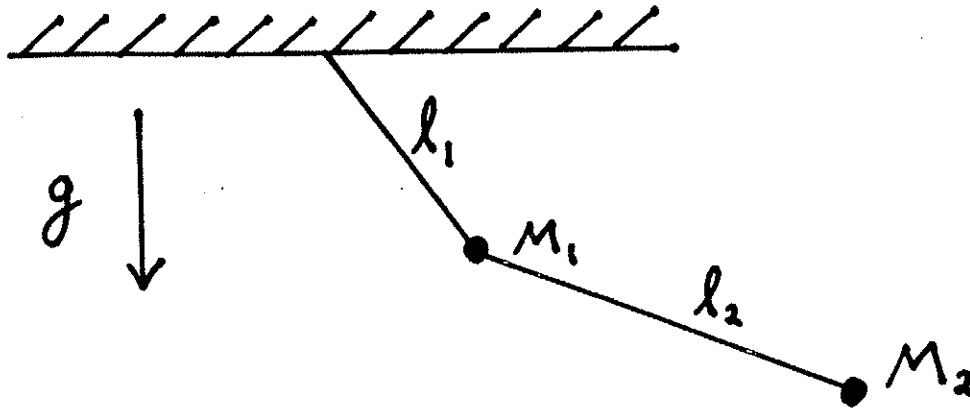
A thin uniform rod of mass  $M$  and length  $2b$  is placed on top of a stationary cylinder of radius  $a$ , so that the center of mass of the rod rests on the uppermost part of the cylinder. The cylinder is attached to the ground and does not move. On both ends of the rod, masses  $m$  are attached by strings of lengths  $l_1$  and  $l_2$ . The setup is then moved from equilibrium to perform small oscillations. The contact between the cylinder and the rod is assumed to be dominated by friction, i.e. the rod does not slide over the cylinder's surface.

- Determine the Cartesian position of the center of mass of the rod as a function of its tilt angle  $\theta$ . Also determine the Cartesian positions of the two masses as a function of the angles  $\theta_1$  and  $\theta_2$  of their respective strings. [3 pts]
- Determine the Cartesian velocities of the three locations in part a) as a function of  $\theta$ ,  $\theta_1$  and  $\theta_2$ . [2 pts]
- Perform a small angle approximation of the velocities and determine the Lagrangian under this approximation. [3 pts]
- Determine the resulting equations of motion and the frequencies of oscillations. [2 pts]



**Problem 1** 10 points

A coplanar double pendulum is constructed of two masses,  $M_1$  and  $M_2$ , and two massless strings of respective lengths  $\ell_1$  and  $\ell_2$ , as indicated; the motion is in a plane.

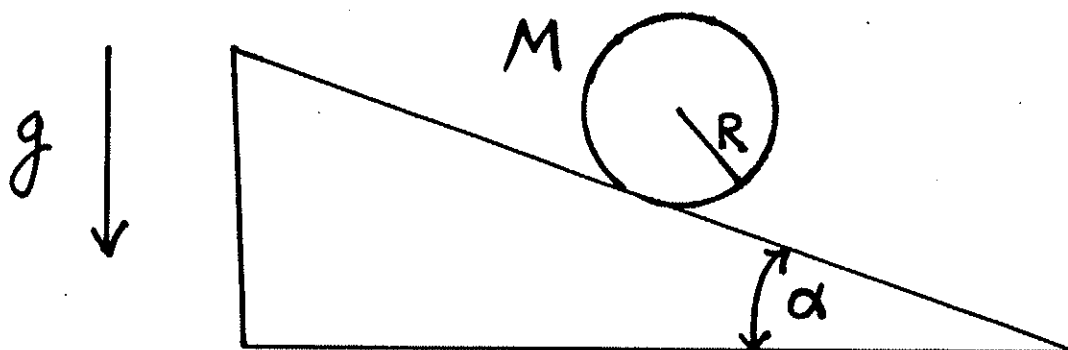


The double pendulum is suspended from a fixed point in a uniform gravitational field  $g$ .

- 3 points      a) Write down a Lagrangian in terms of suitable generalized coordinates.
- 5 points      b) Derive the normal mode frequencies for small oscillations of the double pendulum.
- 2 points      c) Draw pictures which illustrate the normal mode vibrations.

**Problem 2** 10 points

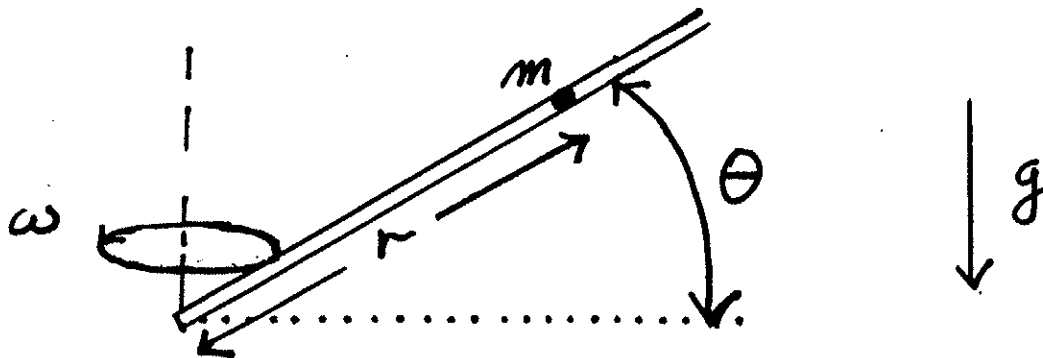
A uniform disk of mass  $M$  and radius  $R$  rolls without slipping on an inclined plane that makes an angle  $\alpha$  with the horizontal. The gravity is  $g$ .



- 1 point      a) Calculate the principle moment of inertia of the disk about its symmetry axis.
- 5 points      b) Using the method of Lagrange's undetermined multipliers  $\lambda$ , find Lagrange's Equations, including the multiplier for appropriate generalized coordinates and choice of Lagrangian.
- 4 points      c) Solve your Lagrange's Equations for arbitrary initial conditions, including the force of constraint provided by the force of friction.

**Problem 3** 10 points

A mass  $m$  slides without friction inside a tube whose length is inclined at an angle  $\theta$  with the horizontal. The tube has one end at a fixed place, and the other end rotates with constant angular velocity  $\omega$  about a vertical line through the fixed end. Let  $r$  be the distance from  $m$  to the end of the tube that is fixed in place. The vertical gravitational field is  $g$ .



- 3 points      a) Write down a Lagrangian  $L(r, \dot{r})$ .
- 4 points      b) Derive Lagrange's Equation from the Lagrangian.
- 3 points      c) Determine  $r(t)$  for the initial values  $r(0)=r_0, \dot{r}(0)=0$  from the equation of motion derived in b).

**Problem 4** 10 points

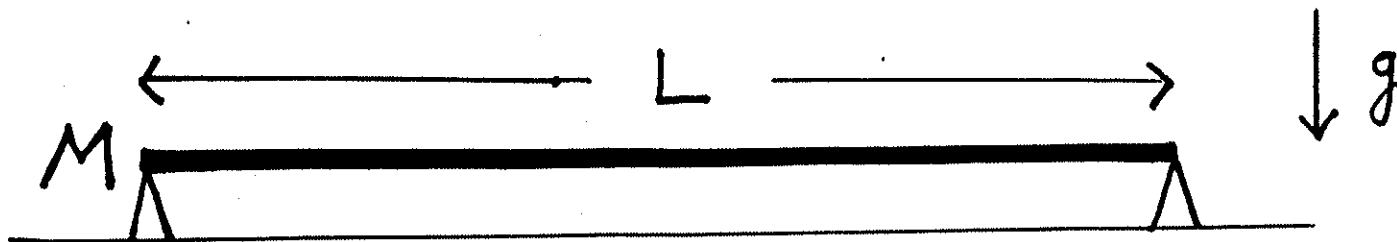
A satellite is put into a circular orbit at a distance  $R_0$  above the center of the earth. The satellite has mass  $m$  and the earth has mass  $M \gg m$ . A viscous drag force resulting from the thin upper atmosphere has the velocity dependence  $|\mathbf{F}_v| = \lambda V^\alpha$ , where  $V = |\mathbf{V}|$  is the instantaneous speed of the satellite. [The viscous drag force is of course directed opposite to the velocity vector  $\mathbf{V}$ . It is observed that this results in a rate of change in the radial distance  $r$  given by  $dr/dt = -C$ , where  $C$  is a positive constant, sufficiently small so that the loss of energy per orbit is small compared to the total kinetic energy.

5 points      a) Derive an expression for  $A$ .

5 points      b) Derive an expression for  $\alpha$ .

**Problem 5** 10 points

A uniform thin rigid rod of mass  $M$  and length  $L$  is initially supported horizontally by two vertical supports at both its ends:



At  $t = 0$  one of these supports is kicked out.

2 points      a) Calculate the moment of inertia of the rod about an end.

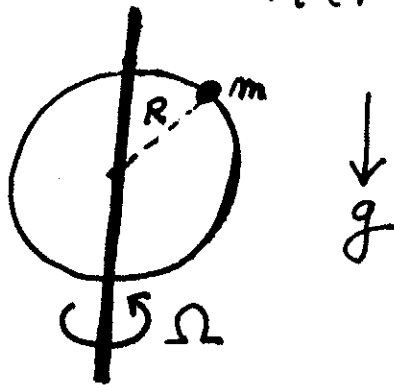
8 points      b) Calculate the force on the other support immediately after the first support is kicked out.

HINT: Use the elementary versions of the general rigid body equations of motion developed in class.

## EXAM I

1. By the use of principles from the calculus of variations, prove that the plane curve in the  $x$ - $y$  plane with the shortest length from the origin to the point  $x=5$ ,  $y=17$  is a straight line, and give its equation. (5 points)

2. A circular wire of radius  $R$  rotates with constant angular velocity  $\Omega$  about a vertical diameter in the presence of gravity  $g$ . A small bead of mass  $m$  is free to move without friction on the circumference, as shown:





(a) Choose independent coordinates for the bead, and write down a Lagrangian.

(2 points)

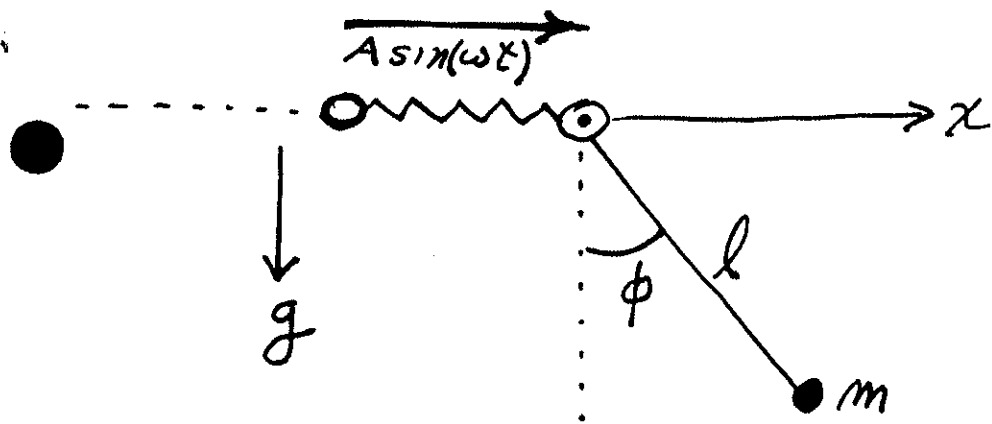
(b) Write down Lagrange's Equations (of the second kind), and derive the equation of motion for the bead.

(1 point)

(c) There are three locations on the rotating circle where the bead will remain, if initially placed there with zero speed along the wire. Specify these three locations.

(2 points)

3. The suspension point of a simple pendulum consisting of a mass  $m$  at the end of a weightless rod of length  $l$  is forced to oscillate horizontally with amplitude  $A$  and given angular frequency  $\omega$ ,  $x = A \sin(\omega t)$  as shown: The gravity is  $g$ .



(a) Choose independent coordinate(s) to describe the system configuration, and write down a Lagrangian.

(2 points)

(b) Derive Lagrange's Equation(s) of motion (2nd kind) for your choice of independent coordinate(s) and Lagrangian.

(1 point)

(c) Show from your equation of motion obtained in part (b) that in the approximation of small angular displacement  $\phi$ , the oscillatory steady state vibration of the pendulum may become large if  $\omega \approx \sqrt{g/l}$ .

(2 points)

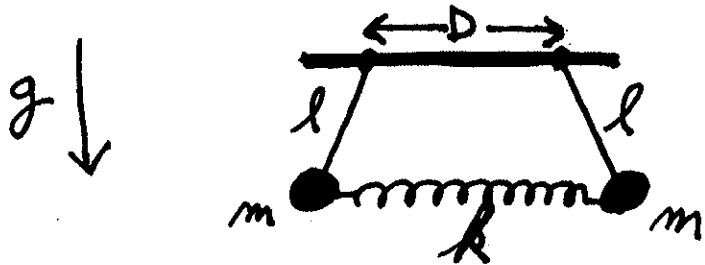
## EXAM 2

- (5 points) An unstable primary particle of mass  $M$  moving with speed  $V$  in the Lab. coordinate frame (L system) has an internal energy  $E_i$  in its own frame of reference (Center of mass system). It disintegrates into three (3) daughter particles with masses  $m_1, m_2, m_3$ , and with disintegration energy  $E$ .

For  $m_1 = M/5$ ,  $m_2 = M/10$ ,  $m_3 = 7M/10$  calculate the MAXIMUM POSSIBLE

Lab. frame energies for EACH ONE of the three daughter particles.

2. (5 points) Consider two coupled simple pendula, each of equal length  $l$  and mass  $m$ , as indicated in the picture.



The masses are coupled by a massless spring whose force constant is  $k$ .

The potential energy of coupling is  $\frac{1}{2}kx^2$  when the masses are separated by an additional coordinate difference  $x$  in any configuration that departs from their equilibrium separation  $D$ .

- D. 3 points a) Calculate the small oscillation eigen frequencies.  
2 points b) Find and display the normal modes.

3. (5 points)

Consider a particle scattered by a fixed attractive central potential  $U(r)$  centered at  $r = 0$ .

Recall that the presence of the finite centrifugal energy when the angular momentum is finite generally - BUT NOT ALWAYS - prevents an incident particle from reaching the origin at  $r = 0$  unless

**CERTAIN CONDITIONS ARE MET.**

If  $U(r) = -\alpha/r^2$ , calculate

the effective cross-section  $\sigma$

for the specific process only in which an incident particle reaches the origin when it

had an initial incident speed  $V_\infty$  at infinite distance.

**Classical Mechanics Graduate Exam**  
December 15, 2004

NOTE: Do NOT write your name on any paper you hand in. Only write the selected number for the subsequent identification, so that the problems may be graded anonymously.

Show all work. Justify your answers.

1. At relative distances  $r > R$ , particles  $A$  of mass  $m_1$  are repelled from particles  $B$  of mass  $m_2$  with central forces  $F = k/r^{5/2}$ . At distances  $r < R$ , inelastic processes set in, altering intrinsic structure of the particles  $A$  and  $B$ .
  - (a) Find the potential  $V$  for the central force  $F$ .
  - (b) Particle  $A$  is directed from far away towards particle  $B$  at rest. What is the minimum initial momentum  $p_1$  of  $A$  needed for the inelastic processes to set in?
  - (c) Find the cross section for inelastic processes, as a function of  $p_1$ .
  
2. (a) Demonstrate that the transformation

$$\begin{aligned} x &= X \cos \lambda + P_Y \sin \lambda, & y &= Y \cos \lambda + P_X \sin \lambda, \\ p_x &= P_X \cos \lambda - Y \sin \lambda, & p_y &= P_Y \cos \lambda - X \sin \lambda, \end{aligned}$$

where  $\lambda$  is an arbitrary parameter, is canonical.

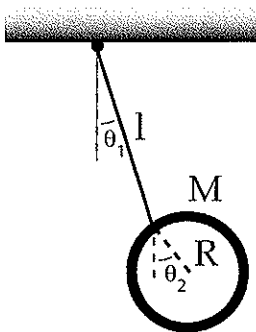
- (b) Assuming the Hamiltonian of a two-dimensional harmonic oscillator:

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2},$$

find the Hamiltonian after the transformation (a), in terms of  $X, Y, P_X$  and  $P_Y$ .

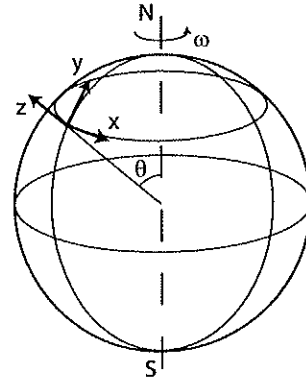
- (c) What is the Hamiltonian for a particle of mass  $m = 1$  and charge  $q = 1$ , confined to the  $x$ - $y$  plane, moving under combined influence of the harmonic oscillator potential  $V = (x^2 + y^2)/2$  and the uniform magnetic field  $\vec{B}$  in  $z$ -direction, represented by the vector potential  $\vec{A} = (0, Bx, 0)$ ?
- (d) Employ the transformation from (a) to the Hamiltonian from (c) and find the condition on  $\lambda$  under which the transformed Hamiltonian becomes that of an anisotropic harmonic oscillator.

3. A thin ring of mass  $M$  and radius  $R$  hangs from a rigid support by a thin thread of length  $\ell$ .

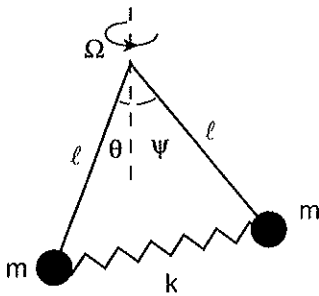


- (a) Write the kinetic and potential energies for the motion of the ring within the plane of the figure.
- (b) Expand the kinetic and potential energies in  $\theta_1$  and  $\theta_2$  by considering small deviations of the ring from its equilibrium position.
- (c) Find the normal modes of oscillation of the ring within the plane, frequencies and amplitude vectors. Discuss those modes.

4. A particle is projected vertically upward from a location at colatitude  $\theta$  on the Earth's surface. If it is intended that the particle falls back to the ground at a distance no greater than  $d$  from the launch point on the ground, what is the maximum elevation  $h$  above the ground to which the particle can rise? Neglect air resistance and obtain the answer to the leading order in the Earth's angular velocity  $\omega$ . In which geographic direction from the launch point will the particle land?



5. The governor of a steam engine consists of two small balls of mass  $m$  that are mounted on light rods of length  $\ell$ . At their other ends, the rods are attached, through hinges, to a vertical axis. The plane of the rods rotates at constant angular velocity  $\Omega$  about the axis. A spring of spring constant  $k$  connects the two governor's balls. Below, assume that the rods are massless, balls are of negligible size and neutral length of the spring is negligible. Use the angles  $\theta$  and  $\psi$  as generalized coordinates.



- Find the kinetic and potential energies and the Lagrangian for the governor.
- Obtain the Lagrange equations.
- Find equilibrium points for the governor. Note: those points are characterized by symmetry  $\theta = \psi$ .
- Obtain the Hamiltonian for the governor.
- Is the Hamiltonian conserved? Is the energy conserved? Is the angular momentum conserved? Explain why.

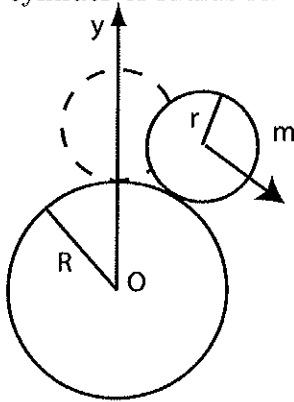
## Classical Mechanics Graduate Exam

August 29, 2005

NOTE: Do NOT write your name on any paper you hand in. Only write the selected number for the subsequent identification, so that the problems may be graded anonymously.

Show all work. Justify your answers.

1. A uniform cylinder of mass  $m$  and radius  $r$  is placed on top of a rigidly held cylinder of radius  $R$ . The axes of the cylinders are parallel.



At time  $t = 0$ , the top cylinder is slightly touched and begins rolling, without slipping, down the bottom cylinder. (a) Find the moment of inertia of a cylinder of mass  $m$  and radius  $r$  about its axial symmetry axis. (b) What is the condition for the top cylinder detaching from the bottom cylinder, during its downward motion? (c) Find the elevation  $y$ , of the top cylinder's center above the bottom cylinder's center, when the top cylinder detaches.

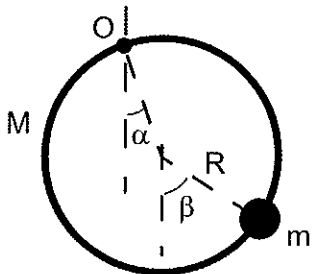
2. Particles 1 and 2, both of mass  $m$ , move along the  $x$ -axis. Particle 1 is affected by an electric trap potential characterized by a constant  $K$ . Particle 2 is neutral and not affected by the trap. The two particles interact with each other via potential characterized by constants  $D$  and  $d$ . The Lagrangian describing this situation is:

$$L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} K x_1^2 - \frac{1}{2} D \left( (x_1 - x_2)^2 - d^2 \right)^2 .$$

- (a) Find equilibrium points for the system of the two particles. What type of equilibrium, stable, unstable or neutral, do the respective points represent?  
 (b) Choose a stable equilibrium point characterized by  $x_1 \geq x_2$  and find angular frequencies of the free system vibrations around that point.

(Over)

3. A thin ring of mass  $M$  and radius  $R$  can swing around a nail, through point  $O$ , pinning the ring to a vertical wall. Threaded on and moving relative to the ring is a bead of mass  $m$ . Consider planar motion of the ring and bead, including the effects of gravity, but ignoring friction.



- (a) Find the kinetic energy  $T$  and Lagrangian  $L$  for the ring-bead system, employing the indicated angles  $\alpha$  and  $\beta$  as generalized coordinates. (b) Find the kinetic energy  $T_{CM}$  associated with the motion of the center of mass of the system. Explain the origin of any differences compared to  $T$  in (a).
- (c) Obtain the generalized momenta conjugate to the angles  $\alpha$  and  $\beta$ . Is any of those momenta conserved? Explain why. (d) Obtain the energy function  $h$ , referred also to as Jacobi integral, for the ring-bead system. Is that function conserved? Is the energy function identical to the energy for the system? Explain under what circumstances (i) the energy function is conserved and (ii) the energy function coincides with the energy.
4. The Hamilton's principle implies that the Lagrangian  $\tilde{L}$ , obtained by adding the time-derivative of an arbitrary differentiable function  $M$  of time and coordinates to the Lagrangian  $L$ ,

$$\tilde{L} = L + \frac{dM}{dt},$$

yields the same Lagrange equations of motion as  $L$ . Here,  $L = L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$  and  $M = M(q_1, \dots, q_n, t)$ . (a) Let  $\tilde{p}_k$  and  $p_j$  represent the generalized momenta obtained, respectively, from  $\tilde{L}$  and  $L$ . Express  $\tilde{p}_k$  in terms of  $p, q$  and  $t$ . (b) Consider the transformation  $(p, q) \rightarrow (\tilde{p}, \tilde{q})$ , where  $\tilde{q}_k = q_k$  and  $\tilde{p}_k = \tilde{p}_k(q, p, t)$  is the relation from (a). For that transformation of the canonical variables, find how the fundamental Poisson brackets (of the canonical variables) transform. (c) Express the difference between the Hamiltonians  $\tilde{H}$  and  $H$ , obtained from  $\tilde{L}$  and  $L$ , respectively, in terms of  $p, q$  and  $t$ . (d) Construct a generating function  $F(q, \tilde{p})$  for the transformation  $(p, q) \rightarrow (\tilde{p}, \tilde{q})$ . Verify that your function is consistent with the results in (a) and (c) and with  $\tilde{q}_k = q_k$ .



## Classical Mechanics Graduate Exam

August 29, 2006

NOTE: Do NOT write your name on any paper you hand in. Only write your selected number for subsequent identification, so that the problems may be graded anonymously. Each problem will be handed in separately, so your number and the problem number should be given on the first page of your work for each problem.

Show all work. Justify your answers.

1. (10 points) In Galileo's famous experiment, two balls of differing radii were dropped from the tower of Pisa. Suppose the balls were iron spheres of density  $\rho_{\text{Fe}} = 7500 \text{ kg/m}^3$  and radii of 1 cm and 5 cm. In addition to the gravitational force, the balls experience a frictional force  $F_{\text{fric}} = C(rv)^2$ , where  $v$  is the speed of the ball,  $r$  is its radius and  $C = 1 \text{ kg/m}^3$ . Assume the balls were dropped simultaneously from a height of 15 m.

(a) (2 pt) Calculate the terminal velocities,  $v_t$ , of the two balls in m/s.

(b) (4 pt) Calculate the velocity of a ball of radius  $r$  as a function of time including the frictional force. Note:

$$\int dx \frac{1}{x^2 - 1} = -\tanh^{-1}(x).$$

(c) (2 pt) Find an approximate expression for the time required for a ball of radius  $r$  to reach a speed of  $0.99v_t$ .

(d) (2 pt) Which ball reaches the ground first? State your reason.

2. (10 points) The function  $y(x)$  which minimizes a variational integral of the form

$$I = \int_{x_0}^{x_1} dx F(y(x), y'(x))$$

where the prime denotes a derivative with respect to  $x$ , satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0.$$

- (a) (3 pt) Using the equations of motion, show that the function  $H$  defined by

$$H = y' \frac{\partial F}{\partial y'} - F$$

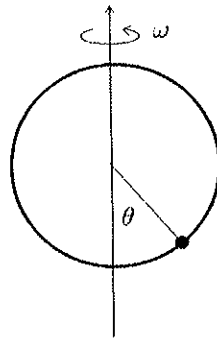
satisfies

$$\frac{dH}{dx} = 0 \quad \text{or} \quad H = \text{constant}.$$

- (b) (7 pt) Using  $H$ , find the function  $y(x)$  which connects the points  $(1, -1)$  and  $(1, 1)$  and minimizes the integral

$$I = \int_{-1}^1 dx \sqrt{y} \sqrt{1 + y'^2}.$$

3. (10 points) A bead of mass  $m$  is constrained to move on a circular hoop of radius  $R$ . The hoop rotates with constant angular velocity  $\omega$  around a fixed vertical diameter of the hoop, parallel to the uniform gravitational field.
- (2 pt) Calculate the Lagrangian for this system in appropriate variables.
  - (1 pt) Derive Lagrange's equations of motion.
  - (2 pt) Derive the Hamiltonian function of canonical variables from its definition in terms of the Lagrangian (explain the steps).
  - (1 pt) Derive Hamilton's equation of motion for the canonical variables.
  - (2 pt) Find the critical angular velocity  $\Omega$  below which the bottom of the hoop is a stable equilibrium position for the bead.
  - (2 pt) Find the stable equilibrium position for  $\omega > \Omega$ .



4. (10 points) Consider the Lagrangian

$$L = (m/2)([dx/dt]^2 - \omega^2 x^2)e^{\gamma t}$$

for motion in the  $x$  direction of a particle of mass  $m$ .

- (a) (2 pt) Derive Lagrange's equation of motion.
- (b) (1 pt) Interpret the equation of motion by stating what kinds of force are acting on the particle.
- (c) (2 pt) Calculate the canonical momentum, and from this construct the Hamiltonian function defined in terms of the Lagrangian.
- (d) (2 pt)
  - i. Is the Hamiltonian a constant of the motion?
  - ii. Is the energy conserved?
  - iii. Is the frictional force doing positive or negative work on the particle?
  - iv. Explain your answers to i), ii) and iii) in words.
- (e) (3 pt) For the initial conditions  $x(0) = 0$  and  $(dx/dt)|_{t=0} = v_0$ , what is  $x(t)$  asymptotically as  $t \rightarrow \infty$ . Find all possible solutions, depending on how the relative magnitudes of  $\gamma$  and  $\omega$  affect the motion.

5. (10 points) Discuss the 2-dimensional motion of a particle moving in an attractive central-force described by the force law  $f(r) = -k/r^\alpha$ , where  $k$  is positive and  $3 > \alpha > 2$ .
- (a) (4 pt) Write down the equations of motion in polar coordinates.
  - (b) (4 pt) Show how conservation laws can be used to derive the formal equation for the orbit of motion.
  - (c) (2 pt) Describe the nature of the orbits for various possible initial energies and angular momenta using a graph of energy vs  $r$ .

**PHY-820: CLASSICAL MECHANICS**  
**FINAL EXAM**  
*December 11, 2008*

**PROBLEM 1.** /20/ a. Find the capture cross section for particles of mass  $m_1$  moving with velocity  $v$  to the surface of the spherical body of radius  $R$  and mass  $M$  that attracts the particles by the gravitational force. Particles are considered captured if they fall on the surface of the big body.

b. Under what conditions is this cross section equal to  $\pi R^2$ ?

**PROBLEM 2.** /20/ Determine the motion  $x(t)$  of a particle of mass  $m$  with energy  $E = 0$  in the potential

$$U(x) = -Ax^4, \quad A > 0. \quad (1)$$

Consider the initial condition  $x(0) > 0$  and two cases,  $\dot{x}(0) > 0$  and  $\dot{x}(0) < 0$ .

**PROBLEM 3.** /20/ An atom of mass  $m$  moving with velocity  $\mathbf{v}$  undergoes an elastic head-on collision with one of the atoms of a diatomic molecule. The molecule consists of two atoms of mass  $m/2$  each and can be modeled by a dumb-bell with distance  $a$  between the atoms. Before the collision, the molecule is at rest being oriented perpendicular to  $\mathbf{v}$  and not rotating. Consider the atoms as massive point-like objects. For the situation after the collision determine:

- a. the velocity  $\mathbf{v}'$  of the first atom,
- b. translational velocity  $\mathbf{u}$  of the molecule,
- c. angular momentum  $L$  of the molecule, and
- d. angular velocity  $\Omega$  of rotation of the molecule.

**PROBLEM 4.** /15/ Two equal charges  $Q$  are fixed at the ends of a vertical line of length  $b$ . A charged bead of mass  $m$  and unknown charge  $q$  of the same sign as  $Q$  can slide along the line.

a. Write down the Lagrange function for the bead and derive the equation of motion (including the gravitational potential).

b. The position of equilibrium of the bead was measured to be at the height  $x_0$  from the lower end of the line. Determine the charge  $q$ .

c. Find the frequency of small oscillations of the bead around the equilibrium.

d. Find the solution  $x(t)$  for small oscillations started at rest at  $t = 0$  with the initial velocity  $v_0$  directed up the line.

**PROBLEM 5. FAST QUESTIONS.** /25/

1. Find the Poisson bracket  $\{\ell_i, (\mathbf{r} \cdot \mathbf{p})\}$ , where  $\mathbf{r}$  and  $\mathbf{p}$  are the radius-vector and the momentum of a particle, respectively, and  $\ell_i$  is the  $i^{\text{th}}$  component of the angular momentum.

2. A charged particle is moving in the static field of two fixed point-like charges. List the constants of motion.
3. A particle is freely moving inside a sphere of radius  $R$  being elastically reflected from the surface of the sphere. How does the energy of the particle change if the sphere is slowly expanding?
4. Two particles with masses  $m$  and  $m'$  are moving along identical trajectories in the same potential field. What is the relation between the times of motion of the particles between given points?
5. Enumerate normal modes of small oscillations of a linear molecule  $\text{CO}_2$ ; in the equilibrium, distances  $\text{OC}$  and  $\text{CO}$  are equal.



# PHY-820 - CLASSICAL MECHANICS

## Final Exam

### Problem 1 - 10 Points



Consider your desired grade dangling in front of you as shown above. The letter is made up of six identical thin rods that are attached rigidly to each other. Each of the six rods has mass  $m$  and length  $l$ . The letter is suspended from the midpoint of the top rod and is free to swing left and right.

- a) [2pts] Determine the center of mass of the letter.
- b) [2pts] Determine the moment of inertia of the letter around the suspension point.
- c) [2pts] Set up the Lagrangian of the system using the generalized coordinate of the angle  $\theta$  of the letter with the vertical.
- d) [2pts] Derive the equations of motion and the stationary point.
- e) [2pts] Linearize the equations of motion around the stationary point and find the frequency of small oscillations.

## PHY-820 - CLASSICAL MECHANICS

### Final Exam

#### Problem 2 - 10 Points

Consider a roller coaster in which a car of mass  $m$  is attached to a frictionless three-dimensional track. The track has a circular footprint with radius  $R$ , i.e. the  $x$  and  $y$  coordinates defining the horizontal plane satisfy  $x^2 + y^2 = R^2$ . The vertical  $z$  coordinate of the track is made to depend on the azimuthal angle  $\theta$ , measured in radians, as  $z = h \cdot (1 + \sin \theta)$ .

- a) [2pts] Determine the Lagrangian  $L$  in terms of  $\theta$  and  $\dot{\theta}$ .
- b) [2pts] Derive the equations of motion.
- c) [2pts] Determine the generalized momentum  $p_\theta$ , and express  $\dot{\theta}$  by  $p_\theta$ .
- d) [2pts] Determine the Hamiltonian  $H$  of the system.
- e) [2pts] Are  $p_\theta$  and/or the Hamiltonian  $H$  conserved, why or why not?

## PHY-820 - CLASSICAL MECHANICS

### Final Exam

#### Problem 3 - 10 Points

Consider a hollow spherical shell of radius  $R$ . On the inside of the spherical shell, a particle of mass  $m$  is moving without friction under the influence of gravity which is of uniform magnitude and pointing vertically downward.

a) [2pts] Assume that the scale of the total energy  $K$  is such that the particle at rest in the lowest point of the sphere corresponds to  $K = 0$ . What is the minimum total energy  $K_{\min}$  the particle needs to have so as to never detach from the shell, regardless of its orbit?

b) [3pts] Assume that the energy of the particle is greater than  $K_{\min}$ . Determine the Lagrangian of the motion using spherical coordinates  $r, \theta, \varphi$ .

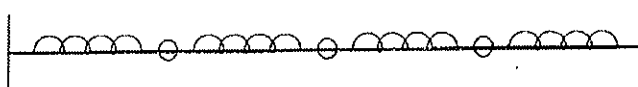
c) [2pts] Determine the generalized momenta  $p_\theta$  and  $p_\varphi$ . Determine which if any of them are conserved, and give a simple explanation.

d) [3pts] Determine the equations of motion.

# PHY-820 - CLASSICAL MECHANICS

## Final Exam

### Problem 4 - 10 Points



Three identical masses  $m$  are sliding without friction on a rod as shown. They are connected with each other and with walls on both ends of the rod through springs of spring constant  $k$  and relaxed length  $l$ . The distance between the two walls is  $4l$ , which leads to an obvious equilibrium position of potential energy 0. As generalized coordinates, use the displacements from the respective equilibrium positions

- [2pts] Determine the kinetic and potential energy and the Lagrangian
- [2pts] Determine the equations of motion, linearize, and express them in matrix form
- [4pts] Determine the eigenmodes and eigenfrequencies of the system
- [2pts] Discuss qualitatively the meaning of the eigenmodes and the associated frequencies, and in particular the relative sizes of the frequencies.

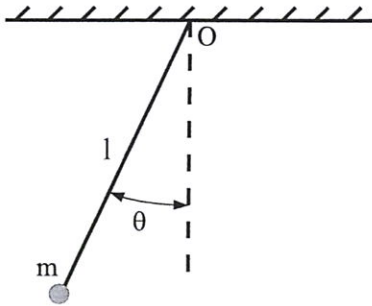
## Classical Mechanics Graduate Exam

August 31, 2004

NOTE: Do NOT write your name on any paper you hand in. Only write the selected number for the subsequent identification, so that the problems may be graded anonymously.

Show all work.

1. Consider a ball of mass  $m$  attached to a massless string of length  $\ell$ . One end of the string is fixed to point  $O$  as shown in the figure. Let the  $z$  axis point downward, let  $\theta$  refer to the angle of the string with respect to the  $z$ -axis and let  $\phi$  represent the azimuthal angle around the  $z$ -axis. The gravitational acceleration is  $g$ .

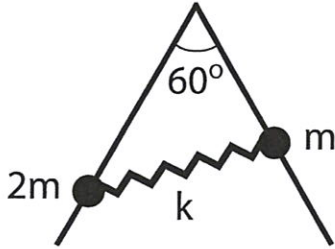


- (a) Write down the Lagrangian, using  $\theta$  and  $\phi$  to represent the position of the ball. (b) Obtain the equations of motion. (c) Obtain the  $z$ -component of the angular momentum in terms of the variables above and show that momentum component is conserved. (d) Find the value of rotational velocity  $d\phi/dt$  required to keep  $\theta$  constant at  $45^\circ$ . (e) The ball, which is initially steadily rotating at  $45^\circ$ , is subjected to a weak jolt in the  $\theta$  direction. Find the frequency at which the ball afterwards oscillates in  $\theta$ , about  $\theta = 45^\circ$ .
2. A particle of mass  $m$  moves in a potential  $V$  that, in spherical coordinates  $(r, \phi, \theta)$ , takes the form

$$V = \frac{A}{r^4} + \frac{B \cos^2 \theta}{r^2}.$$

- (a) Write down a Lagrangian for the particle using spherical coordinates. (b) Obtain the corresponding Hamiltonian in terms of canonical variables. (c) Write down the Hamilton-Jacobi equation for the particle. (d) Solve the Hamilton-Jacobi equation, obtaining Hamilton's principal function in terms of quadratures.

3. Two rods, joined at angle of  $60^\circ$  relative to each other, are installed symmetrically, at an angle of  $30^\circ$  each relative to the vertical. Threaded onto the rods are two beads, of mass  $2m$  and  $m$ , respectively, that can slide along the rods without friction. The beads are interconnected with a spring characterized by a spring constant  $k$  and a negligible unstretched length and mass. (a) Find the equilibrium configuration for the beads. (b) Find the frequencies of normal oscillations of those beads around the equilibrium configuration.



4. (a) The coordinate system  $\mathcal{O}'$  rotates relative to  $\mathcal{O}$  at an angular velocity  $\vec{\omega}$ . How does the rate of change of a vector  $\vec{u}$ , as analyzed in  $\mathcal{O}'$ ,  $\left(\frac{d\vec{u}}{dt}\right)_{\mathcal{O}'}$ , relate to the rate of change in  $\mathcal{O}$ ,  $\left(\frac{d\vec{u}}{dt}\right)_{\mathcal{O}}$ ? (b) Derive the torque-free Euler's equations for the angular-velocity components in the rigid-body frame. (c) Suppose we have a football (or a rugby) ball in the shape of a prolate ellipsoid. The ball's moments of inertia about its principal axes are  $(I_1, I_2, I_3)$ . Assuming  $I_3$  is along the symmetry axis, what is the relation between  $I_1$ ,  $I_2$  and  $I_3$ ? (d) The ball is thrown into space, with an initial angular velocity  $\vec{\omega}$  at an angle  $\alpha$  to the symmetry axis of the ball. Find rate of the  $\vec{\omega}$ -precession in the ball frame.
5. Two beads of mass  $m$  each are connected by a thread of length of  $\pi R/2$ . The beads are hung symmetrically, in an unstable equilibrium, on top of a horizontal cylinder of radius  $R$ . At some instant, one of the beads is gently pulled down and begins to slide down the cylinder side, pulling the other bead behind. (a) What is the condition for a bead to detach from the cylinder surface? Which of the beads, front or rear, detaches first from the surface and why? (b) Find the elevation above cylinder axis where a bead first detaches from the cylinder surface. Ignore thread mass and friction against the surface.

