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MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

Final Exam/Subject Exam

June 7, 2022

Student Number: STUDNUMBER

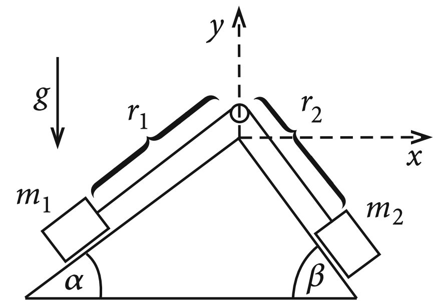
Points					
F1	F2	F3	F4	F5	total
10	10	10	10	10	50

- **PHY422 Students: You only need to complete 3 out of 5 problems.**
- **This is a closed-notes exam: You are not allowed to use the text books, lecture materials or external resources. You are not allowed to discuss this exam or questions related to the exam with your fellow students or with third parties during the exam time window.**
- **Take note of the included formula sheet.**
- **Read through the whole exam before starting to work.**
- **Not all questions are equally difficult, and you may wish to start with problems that play to your strengths.**
- **In some problems, intermediate results are provided as a check and a means to continue working on later parts if you are stuck.**
- **Document all your work (including scratch paper!) so that you can receive partial credit. Justify all your answers!**
- **Do not hesitate to reach out if anything is unclear!**

Good Luck!

Problem F1 – Linked Masses on a Wedge

[10 Points] Two masses m_1 and m_2 that are linked by a rope of length l can slide freely on the sides of a wedge (see figure).



1. State the constraint linking the generalized coordinates r_1 and r_2. Is it holonomic, nonholonomic, or nonholonomic but integrable?

2. Show that the **unconstrained** Lagrangian of the masses is given (up to an irrelevant constant) by

$$L = \frac{1}{2}\left(m_1\dot{r}_1^2 + m_2\dot{r}_2^2\right) + m_1 g r_1 \sin\alpha + m_2 g r_2 \sin\beta. \quad (1)$$

3. Determine the **Lagrange equations of the first kind** and use them to show that the tension in the rope is

$$|\vec{T}| = \frac{m_1 m_2}{m_1 + m_2} g\left(\sin\alpha + \sin\beta\right) . \quad (2)$$

Problem F2 – Central Forces

[10 Points] A particle of mass m is moving in the force field

$$\vec{F}(\vec{r}) = -\frac{3\alpha}{r^4}\vec{e}_r\,, \tag{3}$$

where α is a positive constant.

1. Show explicitly that $\vec{F}(\vec{r})$ is conservative and determine the potential $V(r)$. Choose any potential constant such that $V(r)$ vanishes at large distances.

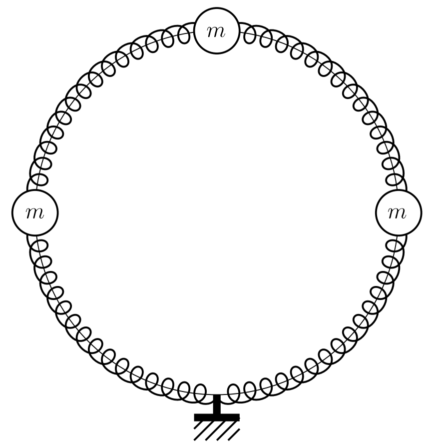
2. Find the zeroes of the effective potential.

3. Determine the radius of circular orbits. Are these orbits stable or unstable?

4. Sketch the effective potential $V_{\text{eff}}(r)$ using the available information about its short- and long-range behavior, the zeroes and the extrema / circular orbits.

Problem F3 – Normal Modes

[10 Points] Three spheres of mass m can slide on a horizontal circular track with radius R and circumference $4l$. As shown in the figure, the spheres are connected by identical massless springs of length l and spring constant k, and the ends of two of the springs are fixed.



1. Construct the Lagrangian in terms of the angular *displacements* ϕ_1, ϕ_2, ϕ_3 of the spheres from their equilibrium positions. Cast it in the usual quadratic form

$$L = \frac{1}{2}\dot{\vec{\phi}} \cdot \boldsymbol{T} \cdot \dot{\vec{\phi}} - \frac{1}{2}\vec{\phi} \cdot \boldsymbol{V} \cdot \vec{\phi} \tag{4}$$

   and show that the matrices $\boldsymbol{T}$ and $\boldsymbol{V}$ are given by

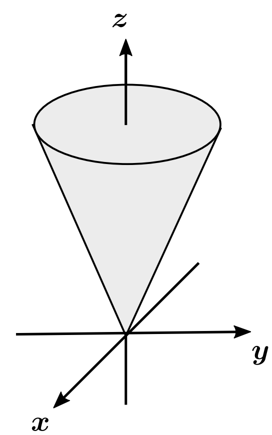
$$\boldsymbol{T} = mR^2\mathbf{1}\,, \quad \boldsymbol{V} = kR^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} . \tag{5}$$

2. Determine the normal modes of the system. **You do not have to normalize the characteristic vectors.**

Problem F4 – Moment of Inertia Tensor of A Cone

[10 Points] Consider a homogenous upside-down cone with base radius R, height H and mass M (see figure). Choosing the tip of the cone as the origin of the coordinate system, its mass density distribution can be parameterized as

$$\rho_M(\vec{r}) = C\Theta(H-z)\Theta(z)\Theta\left(\frac{R}{H}z - \rho\right) \tag{6}$$



1. Compute the volume integral of Eq. (6) in order to show that
$$C = \frac{M}{V} = \frac{3M}{\pi R^2 H}\,. \tag{7}$$
2. Construct the **complete** moment of inertia tensor of the cone for rotations around axes **through its tip**.
3. Determine the cone's center of mass.
4. Find the cone's principal moments of inertia, i.e., the eigenvalues of $\boldsymbol{I}$ in the center-of-mass frame.

Problem F5 – Canonical Transformations of an Oscillator

[10 Points] A harmonic oscillator with mass m and frequency is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \,. \tag{8}$$

1. Consider the coordinate transformation $(q, p) \to (Q, P)$ that is implicitly defined by

   $$q = \frac{P}{m\omega} \sin \frac{m\omega Q}{P} \,, \quad p = P \cos \frac{m\omega Q}{P} \,. \tag{9}$$

   Verify that the transformation is canonical by computing the fundamental Poisson bracket $\{q, p\}_{(Q,P)}$.
   Hint: Assume that $\{Q, P\} = 1$.

2. Construct the Hamiltonian in the new coordinates (Q, P). What kind of system does it describe?

3. Derive the Hamilton equations and give their general solution $Q(t), P(t)$.

4. Revert the canonical transformation to obtain the general solution in the original coordinates.