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MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

Final Exam/Subject Exam

June 1, 2021

Student Number:

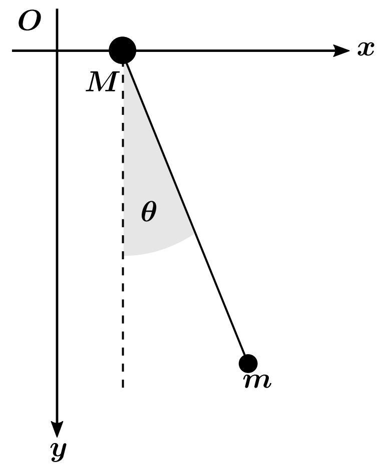
Points					
F1	F2	F3	F4	F5	total
10	10	10	10	10	50

- **PHY422 Students: You only need to complete 3 out of 5 problems.**
- **This is a closed-notes exam: You are not allowed to use the text books, lecture materials or external resources. You are not allowed to discuss this exam or questions related to the exam with your fellow students or with third parties during the exam time window.**
- **Take note of the included formula sheet.**
- **Read through the whole exam before starting to work.**
- **Not all questions are equally difficult, and you may wish to start with problems that play to your strengths.**
- **In some problems, intermediate results are provided as a check and a means to continue working on later parts if you are stuck.**
- **Document all your work (including scratch paper!) so that you can receive partial credit. Justify all your answers!**
- **Do not hesitate to reach out if anything is unclear!**

Good Luck!

Problem F1 – Pendulum with a Moving Suspension

[10 Points] A mass m is suspended via a massless rod of length l from a suspension with mass M that can move freely in x direction. The pendulum is subject to gravity.



1. Construct the Lagrangian of the system.

2. Determine the Lagrange equations.

3. Expand the equations of motion for small displacements θ. What is the oscillation frequency of the pendulum under these conditions? What are the general solutions for $x(t)$ and $\theta(t)$ in this case?

Problem F2 – Central Forces

[10 Points] A particle of mass m is moving in the force field

$$\vec{F}(\vec{r}) = -\frac{nk}{r^{n+1}} \cdot \frac{\vec{r}}{r}\,, \tag{1}$$

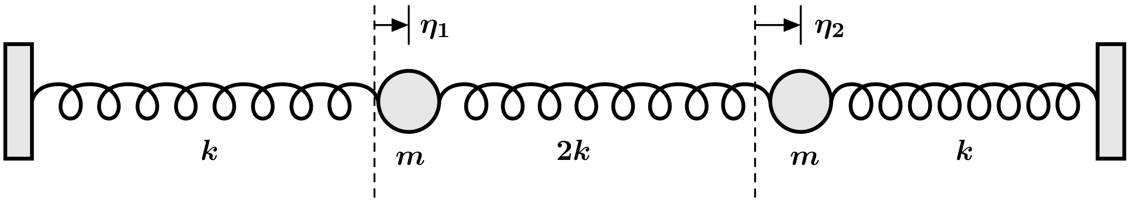
where k is a constant and n a positive integer $(n \in \mathbb{N})$.

1. Show explicitly that $\vec{F}(\vec{r})$ is conservative and determine the potential $V(r)$. Choose any potential constant such that $V(r)$ vanishes at large distances.

2. Under which conditions for k and n can the mass m reach the center of the force field at $r = 0$?

3. Find the radius of circular orbits. For which values of n can these orbits be stable?

Problem F3 – Normal Modes



[10 Points] Consider two identical masses m that are connected to fixed points by springs of strength k, and to each other by a spring with constant $2k$. At rest, the springs have the same length l.

1. Construct the Lagrangian for the system in terms of the displacements η_1 and η_2 of the two masses from their equilibrium positions.

2. Determine the normal modes, i.e., characteristic frequencies and vectors (vectors do not need to be normalized). Sketch your solutions.

Problem F4 – A Physical Pendulum

[**10 Points**] A physical pendulum consisting of thin homogenous rods with masses $M_1 = 2M$, $M_2 = M$ and length $L_1 = L_2 = L$ is swinging under the influence of gravity as shown in the figure.

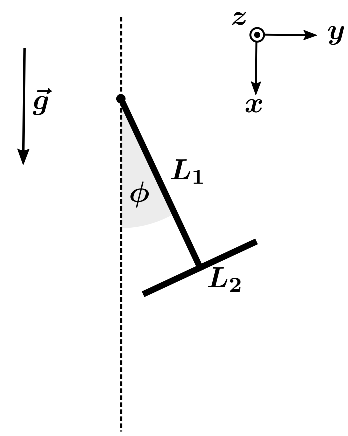
1. Construct the moment-of-inertia tensor of the pendulum, using axes that are oriented as indicated in the figure.
   Note: Prior knowledge can only be used to validate entries of $\boldsymbol{I}$.

2. Determine the pendulum's center of mass.

3. Construct the Lagrangian for the pendulum's motion, using the angle ϕ as the generalized coordinate.

4. Derive the Lagrange equation.

5. Determine the frequency of small oscillations around equilibrium.



Problem F5 – A Charged Oscillator

[10 Points] A particle with mass m and charge $(-e)$ in a constant, homogenous electric field E is described by the following Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 - eEq\,. \tag{2}$$

1. Consider the generating function

$$F(q, P, t) = \left(q - \frac{eE}{m\omega^2} \right) P\,. \tag{3}$$

   Identify the type of F, and use its partial derivatives to determine the coordinate transformation $(q, p) \to (Q, P)$. Verify that the transformation is canonical by computing the fundamental Poisson bracket.

2. Construct the Hamiltonian in the new coordinates (Q, P).

3. Derive the Hamilton equations and give their general solution $Q(t)$.

4. Revert the canonical transformation to obtain the general solution in the original coordinates.