H. Hergert
Facility for Rare Isotope Beams
and Department of Physics & Astronomy

MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

FS 2019

Final Exam

December 19, 2019

Student Number:

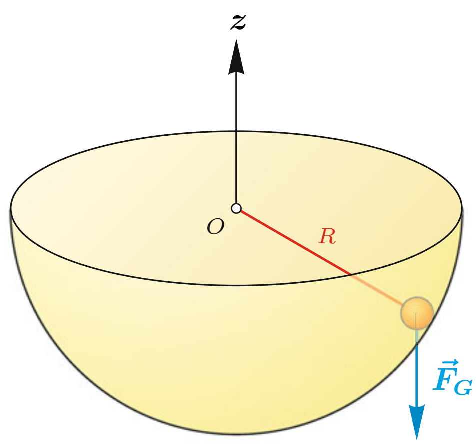
Points					
F1	F2	F3	F4	F5	total
10	10	10	10	10	50

Document your work. Justify all your answers.

Good Luck!

Problem F1 – Spherical Pendulum

[10 Points] Consider a mass m that is suspended from the ceiling by a string of length R, which is swinging under the influence of gravity (see figure).



1. Write down the constraint equation(s).

2. Construct the **unconstrained** Lagrangian. Determine the Lagrange equations of the **first kind** and use them to show that the tension in the string is

$$|\vec{T}| = \left| mR\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - mg\cos\theta \right| . \quad (1)$$

3. Find at least two conserved quantities and either argue or demonstrate explicitly that they are conserved.

Problem F2 – A Central Force

[10 Points] A particle of mass m is moving in the isotropic harmonic oscillator potential

$$V(r) = \frac{1}{2}kr^2\,, \quad k > 0\,. \tag{2}$$

1. Sketch the potential $V(r)$ and the effective potential $V_{\text{eff}}(r)$. Indicate the possible types of trajectories based on their energies E and the turning points (if any exist) of the radial motion.

2. Determine the radius R of circular orbits.

3. Show that near-circular orbits are stable, and prove that the frequency Ω of small oscillations around a circular orbit is twice the oscillator's natural frequency ω:

$$\Omega = 2\omega = 2\sqrt{\frac{k}{m}}\,. \tag{3}$$

Problem F3 – Normal Modes

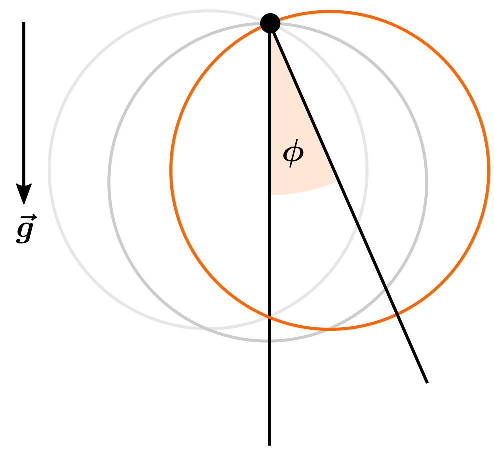
[10 Points] Near equilibrium, a system with three degrees of freedom is described by the Lagrangian

$$L = \frac{1}{2}a\left(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2\right) - \frac{1}{2}b\left(2\eta_1^2 - 4\eta_1\eta_2 + 3\eta_2^2 - 4\eta_2\eta_3 + 4\eta_3^2\right)\,, \tag{4}$$

where η_i denote displacements out of equilibrium, and a, b are real-valued constants. Determine the normal modes of the system. Do the individual modes correspond to linear or oscillatory motion? **Note: You do not need to normalize the characteristic vectors, or sketch the motion.**

Problem F4 – Physical Pendulum

[10 Points] A thin uniform hoop of mass M and radius R is suspended from a nail, and able to swing back and forth in a vertical plane under the influence of gravity (see figure).



1. Compute the hoop's moment of inertia for rotations around the nail.

2. Construct the Lagrangian for the pendulum motion about the nail, using the angle ϕ as the generalized coordinate. Derive the Lagrange equation.

3. Determine the frequency of small oscillations around equilibrium. How does it compare to the frequency of a simple pendulum with mass M and length R, $\omega_{\text{simple}} = \sqrt{g/R}$?

Problem F5 – Hénon-Heiles Hamiltonian

[**10 Points**] Consider a particle of mass m moving in the so-called Hénon-Heiles potential

$$V(x,y) = \frac{1}{2}a\left(x^2+y^2\right) + b\left(x^2y - \frac{1}{3}y^3\right)\,,\quad a,b\in\mathbb{R}\,. \tag{5}$$

1. Construct the Lagrangian and derive the Lagrange equations.
2. Determine the canonical momenta and use them to construct the Hamiltonian for the system.
3. Derive Hamilton's equations and show that they yield the same equations of motion as the Lagrangian formalism.
4. Consider the **special case** $b = 0$, and compute the Poisson bracket $\left\{xp_y - yp_x, H\right\}$. Interpret your result.

   HINT: You can either compute the necessary derivatives directly, or use the relations for Poisson brackets of fundamental variables and product quantities.