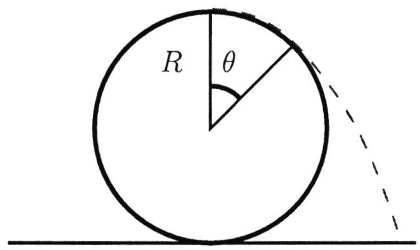
Classical Mechanics Subject Exam

August 28, 2018

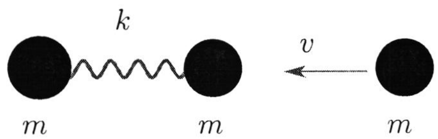
Name: ....

DO NOT WRITE YOUR NAME.
A number is assigned to each graduate student.

1. A heavy frictionless sphere rests on a horizontal plane. A point particle slides down the surface of the sphere starting at the top. Let R be the radius of the sphere. Determine the angle θ that the particle breaks away from the sphere.



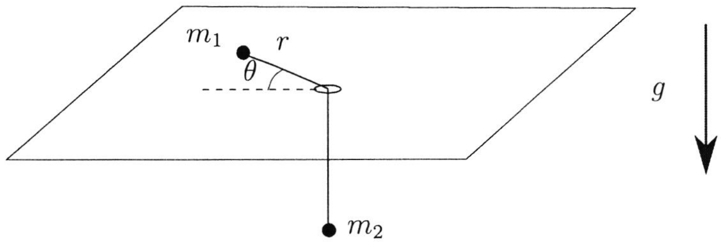
2. A particle collides with a target consisting of two particles attached to a spring that is not initially stretched (see the figure). Assuming that the collision is elastic, find the maximum distance that the spring is compressed after the collision. All particles have the same mass m, the spring constant is k, and the initial velocity of the incoming particle is v.



3. Two mass points m_1 and m_2 are connected by a string of length l passing through a horizontal table. The string and the mass points move without friction with m_1 on the table and m_2 free to move in a vertical line under the action of gravity.

   (a) What initial velocity must m_1 be given so that m_2 will remain mostionless a distance d below the surface of the table?

   (b) For a general motion of the two particles, how many conserved quantities (i.e., integrals of motion) exist? Provide an expression for this or these quantities in terms of the coordinates and their velocities. Argue why this or these quantities are conserved (a mathematical proof is not necessary).

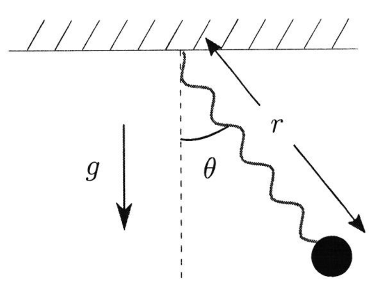


4. A massless spring of unstretched length l_0 and spring constant k is fixed on one end so that it can hang in the gravity while a point mass m is connected to its other end. The motion of the system is only in the vertical plane shown in the figure.

   (a) Write down the Lagrangian in terms of the coordinates (r and θ) shown in the figure.

   (b) Find the Lagrange equations of motion. Determine the equilibrium position of the mass at $\theta = 0$, that is, find the rest length r_0 in equilibrium.

   (c) Expand the equations of motion around the equilibrium position to the lowest order (i.e. assume that $r - r_0$ and θ are small, and expand the equations of motion to the linear order in them). Find the frequency of oscillations around equilibrium. Is the frequency of angular motion different from or the same as that of the radial motion?



5. Consider the Lagrangian for a system with two degrees of freedom x and y:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + m\omega(x\dot{y} - y\dot{x}) \tag{1}$$

(a) Find the equations of motion for both degrees of freedom.

(b) Derive the Hamiltonian $H(x, y, p_x, p_y)$ from the Lagrangian.

(c) Derive the equations of motion from the Hamilton's equations, and show that they are consistent with the Euler-Lagrange equations.

6. Consider a particle of mass m which is constrained to move on the surface of a sphere of radius R. There are no external forces of any kind.

   (a) Choose a set of generalized coordinates and write the Lagrangian of the system. (**Hint:** You can use spherical coordinates.)

   (b) What is the Hamiltonian of the system? Is it conserved?

   (c) Prove that the motion of the particle is along a great circle of the sphere.