General Properties of a Two-body Force

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A general non-relativisit two-body potential V is completely specified by its matrix elements between two-body states, e.g. in the coordinate representation $|\mathbf{r}_1, s_1, t_1; \mathbf{r}_2, s_2, t_2\rangle$; where $s_i = \pm\frac{1}{2}$ and $t_i = \pm\frac{1}{2}$ are spin and isospin coordinates:

$$\langle \mathbf{r}'_1 s'_1 t'_1, \mathbf{r}'_2, s'_2, t'_2 | V | \mathbf{r}_1 s_1 t_1, \mathbf{r}_2, s_2, t_2 \rangle. \tag{1}$$

The space of two-particle states $|\mathbf{r}_1 s_1 t_1, \mathbf{r}_2, s_2, t_2\rangle$ is a product space of the coordinate wave functions $|\mathbf{r}_1\rangle$ and $|\mathbf{r}_2\rangle$ and the spin and isospin vectors $|s_1\rangle$, $|s_2\rangle$ and $|t_1\rangle$, $|t_2\rangle$. Since any operator in the spin space of one particle can be represented as a linear combination of the spin matrices σ_1, σ_2, σ_3 and the unity matrix $\sigma_0 = 1$, the most general form of the operator V is

$$V = \sum_{i,k=0}^{3} V_{ik} \sigma_i^{(1)} \sigma_k^{(2)} \tag{2}$$

The function V_{ik} also depends analogously on the isospin operators $\tau^{(1)}$ and $\tau^{(2)}$. In addition to this isospin dependence, the V_{ik} are, in general, integral operators in coordinate space

$$V|\mathbf{r}_1 \mathbf{r}_2\rangle = \int d^3 r'_1 d^3 r'_2 V(\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}_1, \mathbf{r}_2)|\mathbf{r}'_1, \mathbf{r}'_2\rangle \tag{3}$$

In the special case in which $V(\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}_1, \mathbf{r}_2)$ has the form

$$V(\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}'_1)\delta(\mathbf{r}_2 - \mathbf{r}'_2)V(\mathbf{r}_1, \mathbf{r}_2) \tag{4}$$

V is called a *local potential*, and we have

$$V|\mathbf{r}_1 \mathbf{r}_2\rangle = V(\mathbf{r}_1, \mathbf{r}_2)|\mathbf{r}_1 \mathbf{r}_2\rangle. \tag{5}$$

In this case the interaction between the two particles depends only on the points $\mathbf{r}_1$ and $\mathbf{r}_2$ (and possibly on the spin and isospin). It does not, for instance, depend on the velocity of the particles.

We show now that, in general, nonlocal potentials correspond to a velocity dependence. We therefore expand [1]

$$|\mathbf{r}_1'\mathbf{r}_2'\rangle = |\mathbf{r}_1\mathbf{r}_2\rangle + \sum_{i=1}^{2}(\mathbf{r}_i' - \mathbf{r}_i)\nabla_i|\mathbf{r}_1\mathbf{r}_2\rangle + \ldots$$

$$=: \exp\left\{\sum_{i=1}^{2}(\mathbf{r}_i' - \mathbf{r}_i)\nabla_i\right\} : |\mathbf{r}_1\mathbf{r}_2\rangle \tag{6}$$

and get, from (3),

$$V|\mathbf{r}_1\mathbf{r}_2\rangle = \int \mathrm{d}^3r_1'\mathrm{d}^3r_2'V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2) : \exp\left\{\sum_{i=1}^{2}(\mathbf{r}_i' - \mathbf{r}_i)\nabla_i\right\} : |\mathbf{r}_1, \mathbf{r}_2\rangle$$

$$= \tilde{V}(\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2)|\mathbf{r}_1, \mathbf{r}_2\rangle \tag{7}$$

This means that the most general potential can be represented by Eq. (2) where the quantities V_{ik} are operators in coordinate space of the form (7) (for reasons of simplicity we neglected the isospin dependence).

1 Basic Symmetries of Twobody Forces

In the following we investigate the symmetry properties of such potentials $V(1, 2) = V(\mathbf{r}_1, \mathbf{p}_1, \sigma_1, \tau_1, \mathbf{r}_2, \mathbf{p}_2, \sigma_2, \tau_2)\rangle$. The form of this general function can be restricted by a number of symmetries.

In particular, we require the following eight symmetries:

1. **Hermiticity:**

$$\langle\mathbf{r}_1's_1't_1', \mathbf{r}_2', s_2', t_2'|V|\mathbf{r}_1s_1t_1, \mathbf{r}_2, s_2, t_2\rangle = \langle\mathbf{r}_1s_1t_1, \mathbf{r}_2, s_2, t_2|V|\mathbf{r}_1's_1't_1', \mathbf{r}_2', s_2', t_2'\rangle^*. \tag{8}$$

2. **Invariance under an exchange of the coordinates**

$$V(1, 2) = V(2, 1). \tag{9}$$

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[1] : : means *normal ordering*, i.e., the derivatives ∇_i should not act on the coordinates $\mathbf{r}_i$ in the expansion of the exponent.

φ	χ	abreviation	ζ
even	singlet	es	+
even	triplet	et	-
odd	singlet	os	-
odd	triplet	ot	+

Table 1: Characterization of the symmetries of the two-particle state (10)

This property is strongly connected with the symmetry of the two-particle wave function $|12\rangle$. Since nucleons are fermions, they have to be totally antisymmetric. For example, if we take a product wave function built out of ordinary space, a spin and an isospin part

$$\langle \mathbf{r}_1 s_1 t_1, \mathbf{r}_2 s_2 t_2 | 12 \rangle = \varphi(\mathbf{r}_1, \mathbf{r}_2) \chi(s_1, s_2) \zeta(t_1, t_2) \tag{10}$$

we have four combinations compatible with the Pauli principle, which are characterized by the symmetry of the coordinate space and spin part (Table 1). The isospin component is determined in each case by requiring the antisymmetry of the total wave function (10).

3. **Translational invariance**. The potential depends on the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ only

$$V(1, 2) = V(\mathbf{r}, \mathbf{p}_1, \sigma_1, \tau_1, \mathbf{p}_2, \sigma_2, \tau_2). \tag{11}$$

4. **Galilean invariance**. The potential is not changed by a transformation to a system which moves with constant velocity, that is, it depends only on the relative momentum $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$:

$$V(1, 2) = V(\mathbf{r}, \mathbf{p}, \sigma_1, \tau_1, \sigma_2, \tau_2). \tag{12}$$

5. **Invariance under space reflection**. Contrary to the weak interaction, there is no parity violation for strong interactions:

$$V(\mathbf{r}, \mathbf{p}, \sigma_1, \tau_1, \sigma_2, \tau_2) = V(-\mathbf{r}, -\mathbf{p}, \sigma_1, \tau_1, \sigma_2, \tau_2). \tag{13}$$

6. **Time reversal invariance** guarantees that the equations of motion do not depend on the direction in which the time evolves

$$V(\mathbf{r}, \mathbf{p}, \sigma_1, \tau_1, \sigma_2, \tau_2) = V(\mathbf{r}, -\mathbf{p}, -\sigma_1, \tau_1, -\sigma_2, \tau_2). \tag{14}$$

7. **Rotational invariance in coordinate space**. Rotations in three-dimensional coordinate space act not only on the vectors $\mathbf{r}$ and $\mathbf{p}$ but also on the spin matrices $\sigma = 2\mathbf{s}$. With respect to spin, the operator V has the form (2). It has to be a scalar under a rotation in coordinate space, which means in particular that V_{00} has to be a scalar. There exist three independent scalars which we can construct from the two vectors $\mathbf{r}$ and $\mathbf{p}$, namely $\mathbf{r}^2$, $\mathbf{p}^2$ and $\mathbf{rp} + \mathbf{pr}$. However, the latter expression can only appear quadratically because of time reversal invariance. It is more convenient to express $(\mathbf{rp}+\mathbf{pr})^2$ through $\mathbf{r}^2$, $\mathbf{p}^2$ and $\mathbf{L}^2 = (\mathbf{r}\times\mathbf{p})^2$. V_{00} can then be written as a function of $\mathbf{r}^2$, $\mathbf{p}^2$ and $\mathbf{L}^2$. Because of invariance (ii) and (v) we find

$$V(\mathbf{r}, \mathbf{p}, \sigma_1, \sigma_2) = V(\mathbf{r}, \mathbf{p}, \sigma_2, \sigma_1). \tag{15}$$

The terms in (2) that are linear in $\sigma^{(i)}$ therefore depend only on

$$\mathbf{S} = \frac{1}{2}(\sigma_1 + \sigma_2) \tag{16}$$

To form a scalar, $\mathbf{S}$ has to be multiplied by a vector, which is invariant under space reflection. Only $\mathbf{L}$ fulfils this requirement:

$$\mathbf{LS} = \frac{1}{2}\mathbf{L}(\sigma_1 + \sigma_2) \tag{17}$$

The quadratic terms in σ in Eq. (2) form a tensor. It can be decomposed into a scalar $\sigma_1 \cdot \sigma_2$, a vector $\sigma_1 \times \sigma_2$, and a symmetric tensor with vanishing trace $(\sigma_i^{(1)}\sigma_k^{(2)} + \sigma_k^{(1)}\sigma_i^{(2)})(1 - \frac{1}{3}\delta_{ik})$. Because of (15), $\sigma_1 \times \sigma_2$ cannot appear. As shown by Okubo and Marshak, the only possible independent combinations are

$$\sigma_1\sigma_2, \quad (\mathbf{r}\sigma_1)(\mathbf{r}\sigma_2), \quad (\mathbf{p}\sigma_1)(\mathbf{p}\sigma_2), \quad (\mathbf{L}\sigma_1)(\mathbf{L}\sigma_2) + (\mathbf{L}\sigma_2)(\mathbf{L}\sigma_1). \tag{18}$$

Each of these terms can be multiplied by an arbitrary function of $\mathbf{r}^2$, $\mathbf{p}^2$ and $\mathbf{L}^2$

8. **Rotational invariance in isospin space**. Within the isospin formalism, protons and neutrons are considered as quantum states of one elementary particle that form a doublet with isospin $\frac{1}{2}$. The two-dimensional representation of the rotational group reproduces all their

transformation properties. Rotations within the isospin space (as long as they are not around the 3-axis) produce mixtures of protons and neutrons. Rotational invariance of the nuclear force therefore means the same as charge independence, that is, the proton-proton interaction has the same strength as the neutron-neutron interaction. This has been confirmed by nucleon-nucleon scattering experiments as well as by the symmetry properties of mirror nuclei (e.g., ^{3}He and ^{3}H). Mathematically speaking, this means that the nucleon-nucleon interaction $V(1,2)$ commutes with the operators of the total isospin

$$\mathbf{T} = \mathbf{t}_1 + \mathbf{t}_2. \tag{19}$$

Eigenstates can then be constructed of $\mathbf{T}^2$, T_3 with eigenvalues $T = 0$, $T_3 = 0$ and $T = 1$, $T_3 = -1, 0, +1$ (isospin singlet and isospin triplet). Charge invariance means that $\mathbf{T}^2$ commutes with the operator of the nuclear force. Therefore, the interactions in $T = 1$ states have to be the same (pp, nn, or symmetric pn states). However, they may be different in the $T = 0$ state (antisymmetric pn system). An *isospin invariant*

interaction is there not necessarily *isospin independent*.

The formalism of isospin matrices $\tau = (\tau_1, \tau_2, \tau_3)$ is identical to that of regular spin. There is, however, a difference. Since there is no other vector in isospin space, the only isospin invariant combination of the isospin matrices corresponding to particle l and 2 is

$$V_0 + V_\tau \tau_1 \tau_2 \tag{20}$$

The functions V_0 and V_τ depend on the remaining coordinates such as $\mathbf{r}$, $\mathbf{p}$, σ_1, σ_2, as we have already discussed.

Not all of the combinations possible from the symmetric point of view have been used to describe the nuclear force. We will mention here only the most important terms:

1. Among the local forces, which do not depend on the velocity, the *central force* is the most important. It depends only on the distance r between the nucleons:

$$V_C(1,2) = V_0(r) + V_\sigma(r)\sigma_1\sigma_2 + V_\tau(r)\tau_1\tau_2 + V_{\sigma\tau}(r)\sigma_1\sigma_2\tau_1\tau_2. \tag{21}$$

2. The only remaining local part is the *Tensor force*

$$V_T(1,2) = [V_{T0}(r) + V_{T\tau}(r)\tau_1\tau_2]\, S_{12} \qquad (22)$$

with

$$S_{12} = \frac{3}{r^2}(\sigma_1\mathbf{r})(\sigma_2\mathbf{r}) - \sigma_1\sigma_2 \qquad (23)$$

The term $-\sigma^{(1)}\sigma^{(2)}$ is added to make sure that an average of $V_T(1,2)$ taken over all directions of $\mathbf{r}$ vanishes:

$$\int d\Omega\, V_T(1,2) = 0, \quad \text{where} \quad \mathbf{r} = (r,\Omega). \qquad (24)$$

An experimental hint of the existence of a tensor component in the nucleon-nucleon potential is given by the quadrupole moment of the deuteron, which cannot be explained by pure central forces.

3. The most important non-local term is the *two-body spin orbit interaction*

$$\hat{V}_{LS} = V_{LS}(r)\, \mathbf{L}\mathbf{S}. \qquad (25)$$

In the Hartree-Fock approximation, such a two-body spin orbit potential causes the one-body spin orbit term in the average single-particle nuclear potential, used to explain the magic numbers in nuclei.

4. One sometimes also uses a *second-order spin orbit interaction*:

$$\hat{V}_{LL} = V_{LL}(r)\left\{(\sigma_1\sigma_2)\mathbf{L}^2 - \frac{1}{2}\left[(\mathbf{L}\sigma_1)(\mathbf{L}\sigma_2) + (\mathbf{L}\sigma_2)(\mathbf{L}\sigma_1)\right]\right\}. \qquad (26)$$

2 Special Structures

The central force (21) is the most important part of the nucleon-nucleon interaction. It can also be represented in terms of exchange or projection operators. The operators

$$P^\sigma = \frac{1}{2}(1 + \sigma_1\sigma_2), \qquad P^\tau = \frac{1}{2}(1 + \tau_1\tau_2) \qquad (27)$$

exchange the spin and isospin coordinates, respectively, in a wave function. For instance, we can apply P^σ to the wave function (10) and obtain

$$P^\sigma\varphi(\mathbf{r}_1,\mathbf{r}_2)\chi(s_1,s_2)\zeta(t_1,t_2) = \varphi(\mathbf{r}_1,\mathbf{r}_2)\chi(s_2,s_1)\zeta(t_1,t_2). \qquad (28)$$

This is easy to understand by using the operator (16) of the total spin $\mathbf{S}$. The eigenstates of $\mathbf{S^2}$ are singlet and triplet states and we find:

$$P^\sigma = \frac{1}{2}\left(1 + 2(\mathbf{S}^2 - \mathbf{s}_1 - \mathbf{s}_2)\right) = S(S+1) - 1 = \begin{cases} +1 & \text{for triplet,} \\ -1 & \text{for singlet.} \end{cases} \quad (29)$$

We can also define an operator P^r which exchanges the spacial coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ of the particles. [2] Since the wave function has to be antisymmetric under the exchange of all coordinates of the particles l and 2, the Pauli principle may be written in the form

$$P^r P^\sigma P^\tau = -1. \quad (30)$$

Therefore, in Eq. (21), we can therefore express the products $\sigma_1\sigma_2$ and $\tau_1\tau_2$ by the operators P^σ and P^τ and eliminate P^τ:

$$V_C(1,2) = V_W(r) + V_M(r)\, P^r + V_B(r)\, P^\sigma + V_H(r)\, P^r P^\sigma \quad (31)$$

with the following relations

$$\begin{array}{ll}
V_W = V_0 - V_\sigma - V_\tau + V_{\sigma\tau} & \text{(Wignerforce)} \\
V_M = -4V_{\sigma\tau} & \text{(Majoranaforce)} \\
V_B = 2V_\sigma - 2V_{\sigma\tau} & \text{(Bartlettforce)} \\
V_H = -2V_\tau + 2V_{\sigma\tau} & \text{(Heisenbergforce).}
\end{array} \quad (32)$$

The names of these different components of the nuclear force go back to the years following 1930, when the first models of the nucleus were introduced and the saturation property of nuclear forces was explained by exchange terms without introducing a hard core.

A third way of representing the central force uses projection operators

$$\begin{array}{ll}
\Pi_s^\sigma = \frac{1}{2}(1 - P^\sigma), & \Pi_t^\sigma = \frac{1}{2}(1 + P^\sigma), \\
\Pi_s^\tau = \frac{1}{2}(1 - P^\tau), & \Pi_t^\tau = \frac{1}{2}(1 + P^\tau), \\
\Pi_o^r = \frac{1}{2}(1 - P^r), & \Pi_e^r = \frac{1}{2}(1 + P^r).
\end{array} \quad (33)$$

These are projection operators ($\Pi^2 = \Pi, \Pi^+ = \Pi$), which project onto the singlet (s) and triplet (t), and onto the even (e) and odd (o) part of the

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[2] The operator P^r can be represented by a nonlocal operator in coordinate space, viz:

$$V(\mathbf{r})P^r\varphi(\mathbf{r}_1, \mathbf{r}_2) = \int V(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{r}_1' - \mathbf{r}_2)\delta(\mathbf{r}_2' - \mathbf{r}_1)\varphi(\mathbf{r}_1', \mathbf{r}_2')\mathrm{d}^3 r_1' \mathrm{d}^3 r_2'$$

$\sigma^{(1)}\sigma^{(2)}$	$\tau^{(1)}\tau^{(2)}$	$V(\sigma^{(1)}\sigma^{(2)}, \tau^{(1)}\tau^{(2)})$	$V(P^r, P^\sigma)$	$V(\Pi^r, \Pi^\sigma)$	$|12\rangle$
1	-3	$V_0 + V_\sigma - 3V_\tau - 3V_{\sigma\tau}$	$V_W + V_M + V_B + V_H$	V_{et}	$|et\rangle$
-3	1	$V_0 - 3V_\sigma + V_\tau - 3V_{\sigma\tau}$	$V_W + V_M - V_B - V_H$	V_{es}	$|es\rangle$
1	1	$V_0 + V_\sigma + V_\tau + V_{\sigma\tau}$	$V_W - V_M + V_B - V_H$	V_{ot}	$|ot\rangle$
-3	-3	$V_0 - 3V_\sigma - 3V_\tau + 9V_{\sigma\tau}$	$V_W - V_M - V_B + V_H$	V_{os}	$|os\rangle$

Table 2: Connections between the different representations of a central force.

nuclear two-body wave function (10) in the sense of Table 1. We can express the exchange operators P^σ, P^r by these projection operators and obtain

$$V(1,2) = V_{et}(r)\, \Pi_e^r \Pi_t^\sigma + V_{es}(r)\, \Pi_e^r \Pi_s^\sigma + V_{ot}(r)\, \Pi_o^r \Pi_t^\sigma + V_{os}(r)\, \Pi_o^r \Pi_s^\sigma. \quad (34)$$

Since we have eliminated the isospin exchange operators P^τ in Eq. (31), s and t refers her to spin singlet and spin tripelt. This representation is especially useful in practical applications; for in-stance, in $p - p$ scattering experiments we have only isospin triplet states, (i.e., only es and ot are important). Table 2 shows those functions obtained if one operates with the different representations on wave functions with different symmetry.

The *radial dependence* of the functions V cannot be deduced from invariance principles. In 1937, Yukawa proposed an explanation of the nuclear force using a meson field theory. The nucleons influence each other by the exchange of one or several mesons. The simplest form is the one-pion exchange potential (OPEP). It has the radial dependence of the Yukawa potential:

$$V_Y(r) = \frac{e^{-\mu r}}{\mu r} \quad (35)$$

where $\mu^{-1} = \hbar/\mu_\pi c$ is the Compton wavelength of the pion. The asymptotic form of this potential is uniquely determined by the properties of the pion and its coupling strength to the nucleonic field $f_\pi \simeq 0.08$:

$$\hat{V}_{OPEP} = \frac{f_\pi^2}{4\pi} \frac{m_\pi}{3} \frac{e^{-\mu r}}{\mu r} \tau_1 \tau_2 \left\{ \sigma_1 \sigma_2 + \left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} \right) S_{12} \right\}. \quad (36)$$

A phase shift analysis of the nucleon-nucleon scattering data shows that the OPEP-potential (36) is well able to reproduce the phase shifts for large orbital angular momenta $L \geq 6$. Since these high partial waves only feel the

tail of the nuclear force at large distances ($r \geq 2$ fm), we can assume that the OPEP potential describes the nuclear force properly at such large distances.

The **Reid soft core potential** is also widely used. It has the structure

$$\hat{V} = V_C(\mu r) + V_T(\mu r) + V_{LS}(\mu r)\,\mathbf{LS}. \tag{37}$$

$V_C(x)$ and $V_{LS}(x)$ have the simple form

$$V_C(x) = \sum_{n=1}^{N} a_n \frac{e^{-nx}}{x}, \qquad V_{LS}(x) = \sum_{n=1}^{N} c_n \frac{e^{-nx}}{x}, \tag{38}$$

and $V_T(x)$ is given by

$$V_T(x) = \frac{b_1}{x}\left\{\left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2}\right) - \left(\frac{k}{x} + \frac{1}{x^2}\right)e^{-kx}\right\} + \sum_{n=2}^{N} b_n \frac{e^{-nx}}{x}. \tag{39}$$

The constants are different for all values of T, S and $J \leq 2$. Only a_1, b_1, and c_1 are fixed in such a way that we obtain the OPEP-potential for large distances. For $J > 2$, Reid uses the OPEP-potential.

The Argonne potential v14 has 14 operators:

$$V(1,2) = \sum_{p=1}^{14}(v_\pi^p(r) + v_I^p(r) + v_L^p(r))\hat{O}_{12}^p \tag{40}$$

with the 14 operators $\hat{O}_{12}^p$ are:

$$1, (\sigma_1\sigma_2), S_{12}, \tag{41}$$
$$\mathbf{LS}, \mathbf{L}^2, \mathbf{L}^2(\sigma_1\sigma_2), (\mathbf{LS})^2, \tag{42}$$
$$(\tau_1\tau_2), (\sigma_1\sigma_2)(\tau_1\tau_2), S_{12}(\tau_1\tau_2), \tag{43}$$
$$\mathbf{LS}(\tau_1\tau_2), \mathbf{L}^2(\tau_1\tau_2), \mathbf{L}^2(\sigma_1\sigma_2)(\tau_1\tau_2), (\mathbf{LS})^2(\tau_1\tau_2), \tag{44}$$

As an example we show a few radial functions:

$$v_\pi^{\sigma\tau}(r) = \frac{f_\pi^2}{4\pi}\frac{m_\pi}{3}\frac{e^{-\mu r}}{\mu r}\left(1 - e^{-cr^2}\right), \tag{45}$$

$$v_\pi^{t\tau}(r) = \frac{f_\pi^2}{4\pi}\frac{m_\pi}{3}\left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}\right)\frac{e^{-\mu r}}{\mu r}\left(1 - e^{-cr^2}\right)^2 \tag{46}$$

All in all there are more than 28 phenomenological parameters fitted to the scattering data.