1 Simple exercises (∼ 10-15 minutes maximum for each)

1. Consider a local, spin-independent potential

$$(\mathbf{x}_1\mathbf{x}_2|V|\mathbf{x}_3\mathbf{x}_4) = \delta(\mathbf{x}_1 - \mathbf{x}_3)\delta(\mathbf{x}_2 - \mathbf{x}_4)V(\mathbf{r}_1 - \mathbf{r}_2),$$

   where we are using the shorthand notation $|\mathbf{x}\rangle \equiv |\mathbf{r}, \sigma\rangle$, $\delta(\mathbf{x}_1 - \mathbf{x}_2) \equiv \delta(\mathbf{r}_1 - \mathbf{r}_2)\delta_{\sigma_1\sigma_2}$.

   (a) Find the second quantized expression $\hat{V}$ in terms of field operators $a^\dagger(\mathbf{r}, \sigma)$ and $a(\mathbf{r}, \sigma)$.

   (b) Show that this interaction commutes with the number operator.

   (c) Represent $\hat{V}$ in terms of the momentum space creation/annihilation operators $c^\dagger_{\mathbf{k}\sigma}$ and $c_{\mathbf{k}\sigma}$.

2. Find the second quantized forms for the following operators (shown in "1st quantized" form) for spin 1/2 fermions in terms of the field operators $a^\dagger(\mathbf{r}, \sigma)$ and $a(\mathbf{r}, \sigma)$.

   (a) The matter density operator

$$\rho_N(\mathbf{r}) = \sum_{i=1}^{N} \delta(\mathbf{r} - \hat{\mathbf{r}}_i)$$

   (b) The current density operator

$$\mathbf{j}_N(\mathbf{r}) = \frac{1}{2}\sum_{i=1}^{N}\left\{\frac{\mathbf{p}_i}{m}\delta(\mathbf{r} - \hat{\mathbf{r}}_i) + \delta(\mathbf{r} - \hat{\mathbf{r}}_i)\frac{\mathbf{p}_i}{m}\right\}$$

   (c) The spin density operator

$$\mathbf{s}_N(\mathbf{r}) = \sum_{i=1}^{N} \frac{\sigma_i}{2}\delta(\mathbf{r} - \hat{\mathbf{r}}_i)$$

3. Consider the anti-symmetric jj-coupled two-nucleon (like particles) states

$$|abJM\rangle = N(ab)\sum_{m_a m_b} C(j_a m_a j_b m_b|JM)a^\dagger_{am_a}a^\dagger_{bm_b}|0\rangle,$$

   where $C(j_a m_a j_b m_b|JM)$ is a Clebsch-Gordon coefficient and $a = (n_a, l_a, j_a)$, etc. Determine the normalization constant $N(ab)$. Are there any limitations for the allowable values of J?

4. Same as the previous question, but now work in an isospin representation. I.e., find the normalization of the antisymmetric states $|abJMTM_T\rangle$ and any restrictions on the values of J and T.

2 More involved exercises

1. Consider an infinite homogenous system of spin 1/2 fermions interacting via a local, spin-independent two-body potential.

   (a) Show that the expectation value of the hamiltonian in the non-interacting ground state (i.e., a filled Fermi sea using plane waves in a large box with periodic b.c.'s)

$$\begin{aligned}\langle\Phi|H|\Phi\rangle &= 2\sum_{\mathbf{k}}^{k_F}\frac{\mathbf{k}^2}{2m}+\frac{1}{2}\sum_{\mathbf{kk'}}^{k_F}\sum_{\sigma\sigma'}\langle\mathbf{k}\sigma\mathbf{k}'\sigma'|V|\mathbf{k}\sigma\mathbf{k}'\sigma'\rangle \\ &= 2\sum_{\mathbf{k}}^{k_F}\frac{\mathbf{k}^2}{2m}+\frac{1}{2}\sum_{\mathbf{kk'}}^{k_F}2\,(\mathbf{kk}'|V|\mathbf{kk}')-(\mathbf{kk}'|V|\mathbf{k}'\mathbf{k})\end{aligned}$$

   (b) Now pass to the thermodynamic limit $N\to\infty$, $V\to\infty$, where N/V is finite and constant. Simplify your expression as much as possible.

   (c) Show that if $V(|\mathbf{r}_1-\mathbf{r}_2|)<0$ everywhere and $\int d^3r|V|<\infty$, then the system is unstable to collapse. (Hint: consider the energy per particle as a function of k_F or density.)