Measurement of transverse Emittance

- The beam width x_{max} at s_1 is measured, and therefore $\sigma_{11}(1, k_i) = x^2_{max}(k_i)$.
- Different focusing of the quadrupole $k_1, k_2 ... k_n$ is used: $\Rightarrow \mathbf{R}_{\mathbf{focus}}(k_i)$, including the drift, the transfer matrix is changed $\mathbf{R}(k_i) = \mathbf{R}_{\mathbf{drift}} \cdot \mathbf{R}_{\mathbf{focus}}(k_i)$.
- Task: Calculation of *beam* matrix $\sigma(0)$ at entrance s_0 (size and orientation of ellipse)
- The transformations of the beam matrix are: $\sigma(1, k) = \mathbf{R}(k) \cdot \sigma(0) \cdot \mathbf{R}^{\mathbf{T}}(k)$.

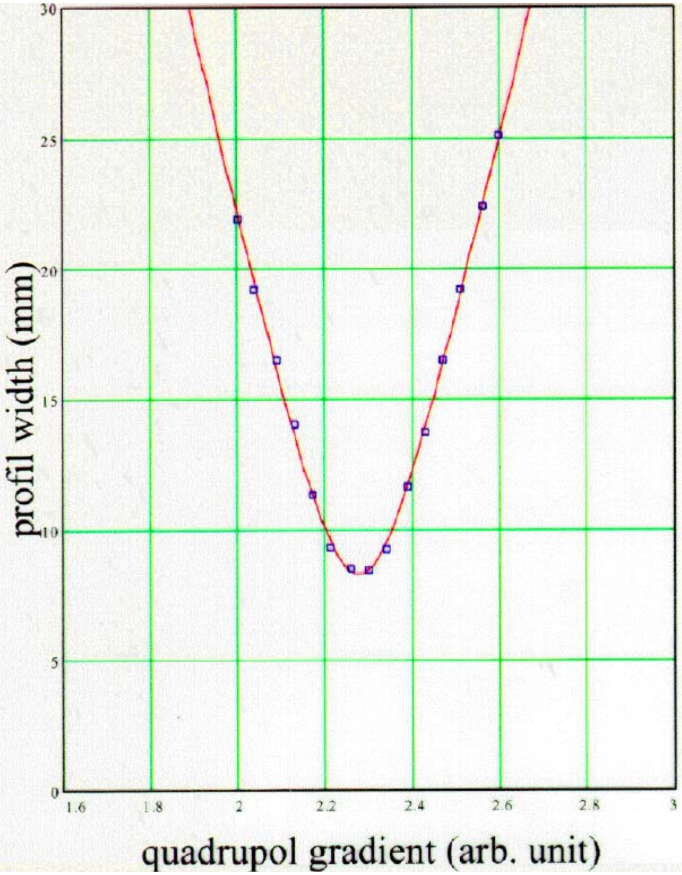
$\Longrightarrow$ Resulting in a redundant system of linear equations for $\sigma_{ij}(0)$:

$$\begin{aligned}
\sigma_{11}(1, k_1) &= R^2_{11}(k_1) \cdot \sigma_{11}(0) + 2R_{11}(k_1)R_{12}(k_1) \cdot \sigma_{12}(0) + R^2_{12}(k_1) \cdot \sigma_{22}(0) \quad \text{focusing } k_1 \\
&\vdots \\
\sigma_{11}(1, k_n) &= R^2_{11}(k_n) \cdot \sigma_{11}(0) + 2R_{11}(k_n)R_{12}(k_n) \cdot \sigma_{12}(0) + R^2_{12}(k_n) \cdot \sigma_{22}(0) \quad \text{focusing } k_n
\end{aligned}$$

- To learn something on possible errors, $n > 3$ settings have to be performed. A setting with a focus close to the SEM-grid should be included to do a good fit.
- *Assumptions:*
  - Only elliptical shaped emittance can be obtained.
  - No broadening of the emittance e.g. due to space-charge forces.
  - If *not* valid: A self-consistent algorithm has to be used.

Measurement of transverse Emittance

Example:
The beam width measured at GSI-LINAC by SEM-grid:



Simplification for 'thin lens approximation':

$$\mathbf{R}_{\mathbf{focus}}(K) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{R}(K) = \mathbf{R}_{\mathbf{drift}} \cdot \mathbf{R}_{\mathbf{focus}} = \begin{pmatrix} 1+LK & L \\ K & 1 \end{pmatrix}.$$

Measurement of $\sigma(1,K) = \mathbf{R}(K)\sigma(0)\mathbf{R}^T(K)$

$$\begin{aligned} \sigma_{11}(1,K) &= L^2\sigma_{11}(0)\cdot K^2 \\ &\quad +2\left(L\sigma_{11}(0) + L^2\sigma_{12}(0)\right)\cdot K + L^2\sigma_{22} + \sigma_{11} \\ &\equiv aK^2 - 2abK + ab^2 + c \end{aligned}$$

The σ-matrix at quadrupole is:

$$\begin{aligned} \sigma_{11}(0) &= \frac{a}{L^2} \\ \sigma_{12}(0) &= -\frac{a}{L^2}\left(\frac{1}{L} + b\right) \\ \sigma_{22}(0) &= \frac{1}{L^2}\left(ab^2 + c + \frac{2ab}{L} + \frac{a}{L^2}\right) \end{aligned}$$

$$\epsilon = \sqrt{\det \sigma(0)} = \sqrt{\sigma_{11}(0)\sigma_{22}(0) - \sigma_{12}^2(0)} = \sqrt{ac}/L^2$$